Endogenous technical change, Spillovers, and Market Structure

Stefan Behringer
http://www.vwl.uni-mannheim.de/gk/_stefan/

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Abstract

This paper investigates the effect of spillovers in a model of endogenous technical change resulting from learning or network effects on the existence of a lower bound to market concentration.

1 Introduction

Why are some industries dominated by very few firms (even on a global scale) while others have only negligible degrees of market concentration? How is this question related to the size of those markets and how do other industry specifics such as learning and network effect affect market structure?

The study of these issues is imperative for all those involved in competition policy analysis as well as for decision making bodies that are concerned with industrial policy in the widest sense. The key statistic in the analysis, *the concentration ratio*, plays a dominant role in the legal and economic assessment of merger cases in front of national and super-national bodies. The $n - firm$ concentration ratio is defined as

$$C_n = \frac{\sum_n x_i}{\sum_N x_i}$$

where the x_i are outputs of the N firms in the industry and the numerator sums the n largest of these. The free availability of data on market concentration makes it a potentially fruitful research area for empirical analysis.[1]

The traditional literature on market structure purports a negative relationship between market size and seller concentration on the grounds that for a given level of 'barriers to entry' an increase in market size should increase profitability of incumbents and thus lead to new entrants. This usually lowers the concentration ratio depending on prior skewness of the size distribution and on what share of a growing market the entrants can capture.[2]

As a reference point for the main analysis we will now briefly discuss what happens in simple *homogeneous good* industries as the market grows. Given that the form of competition will have an effect on the profitability of firms in the industry it will also determine the entry behaviour of firms and hence the concentration ratio of the industry.

In detail this implies that given the structure of the game can be meaningfully represented as having separate periods, if firms face a *weaker* form of competition in the market period(s), more firms will enter the game in the entry period given some total market size, as it is easier for them to recoup an initial sunk entry cost. This last deliberation follows the logic of 'backward induction' commonly used in the game theoretic analysis of stage games.

For competition *à la Cournot* it can be shown (given some mild regularity conditions) that the equilibrium price strictly decreases in the number of firms and approaches the competitive level in the limit (Walrasian equilibrium price) where price distortions will be minimized. This finding is supported by the intuition that as the number of firms increases, each firm's output has a decreasing influence on price and thus firms will act approximately as 'price takers'. Furthermore the equilibrium number of firms that enter the industry will grow unbounded as market size increases. This implies that in large markets the concentration ratio declines to zero.

Given that firms compete *à la Bertrand* in the market period the presence of completely homogeneous products will lead to what is commonly referred to as the 'Bertrand paradox'. It implies that the unique Nash equilibrium of a stage game (in pure strategies) is that firms charge a price equal to marginal cost which has the drastic implication that for any arbitrary small

[1] For some (rather weak) empirical evidence from cross-sectional studies on the market size-concentration ratio relationship see Schmalensee, 1989.

[2] One may think of pathological cases in which entry increases the concentration ratio by the entrant capturing most (or even all) of an increase in market size.

sunk entry cost in the first period at most one firm will enter. Thus the industry will be a monopoly and the concentration ratio equals one for *any* market size.

As is commonly accepted, the Bertrand model provides only an extreme reference for competition since it makes very strong assumptions about the homogeneity of products, the timing, and the form of strategic interdependence. For most (non-Bertrand) specifications of the data however we will find that the concentration ratio will converge to zero as the market grows large.

The side-by-side of Cournot and Bertrand models and their drastic implications given positive setup costs indicate the need for a more refined modelling approach in order to capture equilibrium entry behaviour of firms and the implied development of the concentration ratio in markets that grows out of bounds.

1.1 A condition for non-fragmentation

The above case of complete fragmentation in a homogeneous good market under Cournot competition turns out to be non-robust to additional specifications of the industry in more complex models. A necessary and sufficient condition for non-fragmentation exists: There may be an upper limit on the total number of firms that can profitably enter the industry *independently* of market size.

In order to investigate such an upper bound on the number of firms, Sutton (1998) develops a model of vertically differentiated products with *endogenous technical change* as present in the case of firm specific learning or network effects.[3] He finds that the degree of fragmentation of an industry will be inversely related to the overall strength of these effects. The model is similar to that of Sutton (1991) 'endogenous sunk cost' model which deals with the effects of investments in advertising on market structure. However the equal treatment of the two issues tends to conceal an important difference: In the advertising context, *spillovers* are usually of negligible relevance

[3] The origins of this 'bounds approach' can be traced back much further. See for example Shaked & Sutton (1983) who find an upper bound to the number of firms in a *vertically differentiated* industry as the market gets out of bounds depending on the underlying consumer preferences and the income distribution. They assume that consumers' willingness to pay for quality improvements is an increasing function of income.

as product advertisement may often be well targeted towards a firm's own product. This will not always be possible in a learning or network investment context if firms can free ride on other firms' product innovations.[4]

Sutton's (1998) model abstracts from such complications and predicts that an upper bound on the equilibrium number of firms always exists for any arbitrary small learning effect. He calls this a 'Nonconvergence Theorem' which holds that in the limit the equilibrium concentration ratio will not converge to zero.

The underlying intuition why learning may support a non-convergence result in a stage game with differentiated products and multiple production stages is as follows: Learning implies a form of dynamic increasing returns to scale. Firms will 'overproduce' (above the level that one shot Cournot profit maximization would dictate) in order to be able to 'slide down' the learning curve. This will imply lower equilibrium profits so that initially fewer firms will enter. Learning effects can thus be expected to strengthen the lower bound to concentration by inducing fewer entrants who correctly anticipate the overproduction escalation.

The model investigated here will also look at endogenous technical change in a learning context. In contrast to Sutton (1998) however *we allow for spillovers of learning effects*. We show that, not only do spillovers reduce the effect that learning has on equilibrium market structure but they can even lead to the non-existence of a lower bound to market concentration.

Intuitively this result follows from the fact that the presence of spillovers introduce additional strategic considerations into firms' profit maximizing output decisions. The more learning effects can be used advantageously by the firms' competitors relative to the benefits they imply for the firm itself, the more its incentives to learn are distorted. The overproduction result is weakened leading to larger equilibrium profits so that initially more firms will enter pushing downwards the lower bound to concentration in large markets.

Compared to Sutton's, our model with spillovers predicts that the non-existence of a lower bound to concentration and hence the failure of his 'Non-convergence Theorem' is much more prevalent given that inter-firm spillover effects are large relative to intra-firm learning or that the overall learning effect is small.

[4]With regard to spillovers and market concentration, Fudenberg & Tirole (1983) show that in a model where learning effects have a direct effect on other firms' marginal cost, firms' equilibrium output can be shown to decrease with the degree of learning spillover.

2 The Model

We extend Sutton's (1998) partial equilibrium model with vertically differentiated goods and endogenous technical change by introducing spillovers into the analysis. In the first period firms decide whether or not to enter at some cost. In the second how much of the goods to supply to the market which clears at some equilibrium price p^*. In the third period again supply decisions are taken and the market is cleared. The strategic interdependence is modelled á la Cournot. In each of the two market periods consumers behave non-strategically.

Output decisions in the second period will influence the quality of the product in the third. We analyze the model in a stage game framework with each of the three periods corresponding to a stage. Firms are strategic players and its total payoffs correspond to the sum of second and third stage payoffs. The solution concept is Subgame Perfect Nash equilibrium (SPNE). Most proofs are relegated to the Appendix.

The demand side of the model is as follows. The consumer's utility function is Cobb-Douglas of the form

$$U = (\sum_j u_j x_j)^{\delta} z^{1-\delta} \tag{1}$$

where $x_j \in R_+$ is a 'quality good' produced by firm j (with quality parameter $u_j \in U$ with $U = \{u \,| u \in R_+ \text{ and } u \geq 1\}$), and $z \in R_+$ is an 'outside good' or a composite commodity. Let all consumers have identical incomes.

Suppose that at the end of stage one N firms have entered ($N \in N_+$) which we will call 'active'. They are indexed by $j = 1...N$ and are assumed to produce one quality good in each of periods two and three. Given a price vector $\mathbf{p} = (p_1, ...p_N)'$, $(p_j) \in R_+$ and a quality vector $\mathbf{u} = (u_1, ...u_N)'$ it is well known that a consumer's demand for the product of some firm $k \neq j$ takes the simple form

$$x_k = \begin{cases} 0 \text{ if } \dfrac{p_k}{u_k} > \min\limits_{j \neq k} \dfrac{p_j}{u_j} \end{cases}$$

Thus all quality goods with positive sales in equilibrium must have prices proportional to their qualities

$$p_j = \lambda u_j \ \forall \ j = 1, ...i, ...N. \tag{2}$$

How consumers individually allocate their budget over the quality goods supplied by the firms is indeterminate but we can get a condition for the aggregate of all consumers. Total expenditure on all quality goods of all consumers equals a constant fraction δ, of total consumer income Y. We denote this aggregate level of expenditure by $\delta Y \equiv S \in R_{+}$.

The proportionality factor λ can be found from the aggregate budget constraint which also gives the condition for the vector of *equilibrium market prices.* Total consumer expenditure on all varieties of the quality good must equal total market size S so that

$$\sum_j p_j x_j = \sum_j \lambda u_j x_j = S \tag{3}$$

or

$$\lambda = \frac{S}{\sum_j u_j x_j} \tag{4}$$

Prices for good i are then given as

$$p_i = \frac{S}{\sum_j u_j x_j} u_i \tag{5}$$

using $p_j = \lambda u_j \ \forall \ j$ again and profits for some firm i producing this good given common marginal costs c are

$$\pi_i = (p_i - c)x_i = \left(\frac{S}{\sum_j u_j x_j} u_i - c \right) x_i \tag{6}$$

Proposition 1 *Under the previous assumptions, for any quality vector* $\mathbf{u} = (u_1, ...u_N)'$ *the stage game in period three has a unique Nash equilibrium in which it is firm* i'*s equilibrium strategy to produce output*

$$x_i^* = \frac{S}{c} \frac{N-1}{u_i \sum_{j=1}^{N} \frac{1}{u_j}} \left\{ 1 - \frac{N-1}{u_i \sum_{j=1}^{N} \frac{1}{u_j}} \right\} \tag{7}$$

which implies equilibrium prices of

$$p_i^* = \frac{c u_i}{N-1} \sum_{j=1}^{N} \frac{1}{u_j} \tag{8}$$

and equilibrium net profits given a common marginal cost are

$$S\pi(u_i\,|\mathbf{u}_{-i}) = S\left\{1 - \frac{N-1}{u_i\sum_{j=1}^{N}\frac{1}{u_j}}\right\}^2 \tag{9}$$

Note that $j = 1...i...N$ and $\mathbf{u}_{-i}$ denotes a vector of quality of all other firms of the form $(u_1, ...u_{i-1}, u_{i+1}, ...u_N)'$ with generic element (u_{-i}). Note that as in the standard homogeneous good case the common cost component c falls out in the calculation of equilibrium profits.

2.1 Learning and spillovers

We now introduce learning in this context. We assume that there are *intra-firm learning effects* but also *inter-firm learning spillovers* between the two production stages so that perceived quality may depend on other firms' outputs. The overall degree to which learning in one stage affects future quality is modeled by an elasticity parameter.

The quality level u_i of firm i in stage three is parameterized as

$$u_i = \max(1, \alpha x_i + \boldsymbol{\beta}'\mathbf{x}_{-i})^{\frac{1}{\gamma}}\ \forall\ i = 1...N \tag{10}$$

where γ is an elasticity parameter giving the percentage change in quantity necessary to achieve a unit change in quality, i.e. 'overall' learning effect with $\gamma \to \infty$ implying no learning. α is a scalar that determines the degree of intra-firm learning with $\alpha \in [0, 1]$ and $\boldsymbol{\beta}'$ is a the transpose of a $(N-1)\times 1$ column vector with generic elements $(\beta_i) \in [0, 1]$ that gives the degree of inter-firm learning or spillovers for each firm.

This parameterization allows us to consider the entire $N \times N$ dimensional cube of learning effects rather than only some convex combination of intra- and inter-firm effects. The quality level in stage two is assumed to be unity.

Given that in stage two firms are fully symmetric and we look for a *symmetric* equilibrium of the subgame starting from stage two we know that *on the equilibrium path* all firms will have the same quality level in stage three too. Any one firm deviating will therefore take its rivals to have symmetric qualities.

Equilibrium *stage three net profits of a deviant firm*, given all rival firms have the same quality level (so that $\mathbf{u}_{-i}$ has generic elements $(u_{-i}) = \bar{u}$) depend only on *relative* quality which can be seen by rewriting (9) as

$$S\pi(u_i\,|\mathbf{u}_{-i}) = S\left\{1 - \frac{N-1}{u_i\left[(N-1)\frac{1}{\bar{u}} + \frac{1}{u_i}\right]}\right\}^2 =$$

$$S\left\{1 - \frac{1}{\frac{1}{N-1} + \frac{u_i}{\bar{u}}}\right\}^2 \tag{11}$$

If $\mathbf{u}_i$ has generic elements $(u_i) = \bar{u}\ \forall\ i$ we find that

$$S\pi(\bar{u}_i\,|\bar{\mathbf{u}}_{-i}) = S\pi(\bar{\mathbf{u}}) = \frac{S}{N^2}. \tag{12}$$

As stage three profits in equilibrium depend only on the relative quality levels which in turn depend on the quantity choices in stage two via the learning technology we can calculate the optimal quantity choice as a subgame perfect equilibrium for stage two of the model.

Proposition 2 *Assuming full spillover symmetry ($\beta_i = \beta\ \forall\ i$) and $N > 1$, the subgame for periods two and three with N active firms has a unique symmetric subgame perfect equilibrium. The equilibrium involves period two quantity choices of*

$$x^* = \frac{S}{Nc}(1 - \frac{1}{N}) + \frac{1}{\gamma}\frac{\alpha - \beta}{\alpha + \beta(N-1)}2\left[N + \frac{1}{N} - 2\right]\frac{S}{N^2 c} \tag{13}$$

Proof:

Total profits are given by the sum of stage two and stage three profits as

$$\Pi_i = (p - c)x_i + S\pi(u_i\,|\mathbf{u}_{-i}) \tag{14}$$

which due to the technology (10) depends on stage two quantities only.

Quality symmetry $\bar{\mathbf{u}} = \mathbf{i}$ in stage two implies that all products sell at a common price which we find from (3) as $p = \frac{S}{\mathbf{i}'\mathbf{x}}$ where column vector $\mathbf{i}$ contains a column of $1's$ and $\mathbf{x}$ denotes the total quantity vector.

The optimal stage two *quantity* is derived from the first order condition

$$\frac{\partial \Pi_i}{\partial x_i} - \left[\frac{S}{\mathbf{i'x}} - \frac{S}{\mathbf{i'xx'i}} x_i - c \right] = S \frac{\partial \pi}{\partial u_i} \frac{\partial u_i}{\partial x_i} + S[\nabla \pi(\mathbf{u}_{-i})]' \nabla \mathbf{u}_{-i}(x_i) = 0 \quad (15)$$

where the last term results from the fact that there will be spillovers from firm $i's$ quantity decision in stage two on the other firms' qualities in stage three. These affect firm $i's$ profits adversely. Hence there are *strategic effects* that firm i will take into account when deciding about output in stage two. The assumption of output symmetry at this stage implies that on the equilibrium path firms will also have symmetric qualities in stage three of the game, i.e. $(u_i) = \bar{u} \; \forall \; i$.

The first element on the RHS of the first order condition is

$$\frac{\partial \pi}{\partial u_i} \frac{\partial u_i}{\partial x_i} \Big|_{(u_i)=\bar{u}} = 2 \frac{(N-1)^2}{N} \times \frac{\alpha}{\gamma} \frac{1}{\alpha + \mathbf{i}'\boldsymbol{\beta}} \frac{\pi}{x} \quad (16)$$

the second will be

$$[\nabla \pi(\mathbf{u}_{-i})]' \nabla \mathbf{u}_{-i}(x_i) \Big|_{(u_i)=\bar{u}} = -2 \frac{(N-1)^2}{N} \times \frac{\mathbf{i}'\boldsymbol{\beta}}{\gamma(N-1)} \frac{1}{\alpha + \mathbf{i}'\boldsymbol{\beta}} \frac{\pi}{x} \quad (17)$$

derivations of both equations are relegated to the Appendix.

Inserting (16) and (17) and replacing π by (12) in (15) we find

$$\begin{aligned} \frac{\partial \Pi_i}{\partial x_i} &= \left[\frac{S}{\mathbf{i'x}} (1 - \frac{x_i}{\mathbf{x'i}}) - c \right] + 2 \frac{(N-1)^2}{N} \times \frac{\alpha}{\gamma} \frac{1}{\alpha + \mathbf{i}'\boldsymbol{\beta}} \frac{S}{N^2 x} \\ & -2 \frac{(N-1)^2}{N} \times \frac{\mathbf{i}'\boldsymbol{\beta}}{\gamma(N-1)} \frac{1}{\alpha + \mathbf{i}'\boldsymbol{\beta}} \frac{S}{N^2 x} = 0 \end{aligned} \quad (18)$$

so that the own quantity effect on the average quality of the opponents is $\frac{\mathbf{i}'\beta}{\alpha(N-1)}$ times the effect on the own quality level. In case of full *inter-firm spillover symmetry* ($\beta_i = \beta \; \forall \; i$) the first order condition simplifies to

$$\frac{\partial \Pi_i}{\partial x_i} = \frac{S}{Nx} (1 - \frac{1}{N}) - c + \frac{1}{\gamma} \frac{\alpha - \beta}{\alpha + \beta(N-1)} 2 \left[N + \frac{1}{N} - 2 \right] \frac{S}{N^2 x} = 0 \quad (19)$$

which we can solve for the optimal per firm production in stage two as

$$x^* = \frac{S}{Nc} (1 - \frac{1}{N}) + \frac{1}{\gamma} \frac{\alpha - \beta}{\alpha + \beta(N-1)} 2 \left[N + \frac{1}{N} - 2 \right] \frac{S}{N^2 c}$$

To satisfy the non-negativity constraint for output in case $\alpha < \beta$ we assume that $\gamma \geq \frac{2}{N}$ so that $\gamma_{\alpha<\beta} \in [\frac{2}{N}, \infty)$ and $x^* \in [0, \frac{S}{c}\frac{N+1}{N^2}]$. As none of the following results requires that $\gamma \to 0$ the assumption does not constrain the generality of the analysis. ¥

For the *monopoly case* $N = 1$ the subgame has no equilibrium as the Cobb-Douglas utility function and its iso-elastic demand form imply that the monopolist has profits of $S - cx_m$ so that for any quantity x_m the monopolist can increase profits by reducing x_m where it obtains the same revenue at lower cost. Hence there exists no subgame perfect equilibrium for this case. This problem may be neglected because in our setting as we can always assume that S is large enough to warrant entry of more than one firm.

We will thus focus on the case $N > 1$ in what follows.

Lemma 1 The following comparative statics results hold for $N > 1$

$$\frac{\partial x^*}{\partial \beta} < 0, \frac{\partial x^*}{\partial \alpha} > 0, \frac{\partial x^*}{\partial \gamma} |_{\alpha>\beta} < 0, \frac{\partial x^*}{\partial \gamma} |_{\beta>\alpha} > 0$$

We thus find that larger inter-firm learning spillover and lower intra-firm learning will reduce the subgame perfect equilibrium output in stage two. The presence of spillovers introduce additional strategic considerations into firms' profit maximizing output decisions which can be seen in (15). The more learning effects can be used advantageously by the firm's competitors relative to the benefits they imply for the firm itself, the more its incentives to learn are distorted an hence the less it will decide to produce.

A larger overall learning elasticity parameter γ (lower learning possibilities) given intra-firm learning dominates inter-firm learning spillovers will decrease the subgame perfect equilibrium output in stage two. This follows from the fact that given own learning is less effective the firm will decide to incur less costs (in the form of higher output) to benefit from it. The reverse holds too: If inter-firm learning spillovers dominate individual learning and overall learning becomes less effective the firm will decide to produce more output as its competitors will benefit less strongly from its own investment.

For identical learning parameter values ($\alpha = \beta$) we find from (13) that the equilibrium output being identical to Cournot quantities with homogeneous goods. The same trivially holds for no overall learning effects.

Proposition 3 *Equilibrium profits for firm i in the subgame perfect equilibrium of the above Proposition are*

$$\Pi_i = 2\frac{S}{N^2}\left\{1 - \frac{1}{\gamma}\frac{\alpha-\beta}{\alpha+\beta(N-1)}\left[N + \frac{1}{N} - 2\right]\right\} \tag{20}$$

Proof:

Given (13) stage two profits are

$$\begin{aligned}(p-c)x^* = \; & (\frac{S}{\mathbf{i}'\mathbf{x}} - c)x^* = \frac{S}{N} - cx^* = \\ & \frac{S}{N} - \left[\frac{S}{N}(1-\frac{1}{N}) + \frac{1}{\gamma}\frac{\alpha-\beta}{\alpha+\beta(N-1)}2\left[N+\frac{1}{N}-2\right]\frac{S}{N^2}\right] = \\ & \frac{S}{N^2} - \frac{1}{\gamma}\frac{\alpha-\beta}{\alpha+\beta(N-1)}2\left[N+\frac{1}{N}-2\right]\frac{S}{N^2}\end{aligned} \tag{21}$$

and total profits are (given each firm earns profits $S\pi(\bar{\mathbf{u}}) = \frac{S}{N^2}$ in stage three)

$$\Pi_i = 2\frac{S}{N^2}\left\{1 - \frac{1}{\gamma}\frac{\alpha-\beta}{\alpha+\beta(N-1)}\left[N + \frac{1}{N} - 2\right]\right\} \tag{22}$$

To satisfy the non-negativity constraint for total profit in case $\alpha > \beta$ we assume that $\gamma \geq \frac{(N-1)^2}{N}$ so that $\gamma_{\alpha>\beta} \in [\frac{(N-1)^2}{N}, \infty)$ and $\Pi_i \in [0, \frac{S}{N} + \frac{S}{N^2}]$. ¥

The last condition essentially implies that half the elasticity of perceived quality on profits has to be smaller than the inverse learning elasticity γ, i.e. the change in quantity required for a change in quality. Otherwise the 'cost' of learning is too low relative to the profits form it so that firms produce too much to make positive profits. Note that this condition implies that the previous condition for non-negative quantities $\gamma_{\alpha<\beta} \geq \frac{2}{N}$ always holds.

How do spillovers affect total profits? From the fact that learning possibilities will induce firms to produce quantities in stage two that are above the one shot profit maximizing level under Cournot assumptions to be able to 'slide down' learning curve, more overproduction will decrease total profits further so that we find the opposite comparative statics effects to the previous Lemma 1 for $N > 1$:

$$\frac{\partial\Pi_i}{\partial\beta} > 0, \frac{\partial\Pi_i}{\partial\alpha} < 0, \frac{\partial\Pi_i}{\partial\gamma}\,|_{\alpha>\beta} > 0, \frac{\partial\Pi_i}{\partial\gamma}\,|_{\beta>\alpha} < 0.$$

The derivatives of Lemma 1 imply

$$\frac{\frac{\partial x^*}{\partial \alpha}}{\left|\frac{\partial x^*}{\partial \beta}\right|} = \frac{\left|\frac{\partial \Pi_i}{\partial \alpha}\right|}{\frac{\partial \Pi_i}{\partial \beta}} = \frac{\beta}{\alpha}$$

i.e. the absolute value of a change of α has a proportional effect on profits to a change in β where the proportionality factor is given by the inverse ratio of the two effects. This form of *symmetry* holds *independently* of γ, c or N. This implies that given intra-firm learning α is large relative to inter-firm learning spillovers β a unit change (increase) in β will have a much larger (negative) effect on profits than a unit change (increase) in α. The independence of N is interesting as this implies that this effect is also independent of the total size of the market.

For $\alpha = \beta$ (or $\gamma \to \infty$) we find the Cournot level of profit without any learning effect for the two periods as $\Pi_i^C \left|_{\alpha=\beta}\right. = 2\frac{S}{N^2}$.

Looking at the *corner solutions* for total profits (20) we see that for $\alpha > \beta$ the extreme case $\alpha = 1$, $\beta = 0$ and any $\gamma_{\alpha>\beta} \in [\frac{(N-1)^2}{N}, \infty))$

$$\Pi_i \left|_{\alpha=1,\beta=0}\right. = 2\frac{S}{N^2}(1 - \frac{1}{\gamma}(N + \frac{1}{N} - 2)) < \Pi_i^C \ \forall \ \gamma.$$

yields profits below the Cournot level. Assuming that γ reaches its lower support the firm will make total profits of *zero* producing $x^* \Big|_{\alpha=1,\beta=0,\gamma=\frac{(N-1)^2}{N}} = \frac{S}{c}\frac{N+1}{N}$ from (13). Here firms may borrow and subsidize the learning process in period two with its profits in period three. This may be interpreted as a form of *predatory pricing*. If a *no-borrowing constraint* was imposed the support would become $\gamma_{\alpha>\beta} \in [2\frac{(N-1)^2}{N}, \infty)$.

For $\beta > \alpha$ and any bounded γ one sees from (20) that profits will be larger than under homogenous good Cournot assumptions. Optimal quantities given the *non-negativity constraint on quantities*, $\gamma_{\alpha<\beta} \geq \frac{2}{N}$ binds are $x^* \Big|_{\alpha=0,\beta=1,\gamma=\frac{2}{\gamma}} = 0$. *Maximum profits* are then given as

$$\Pi_i^{\max} \Big|_{\alpha=0,\beta=1,\gamma=\frac{2}{N}} = \frac{S}{N} + \frac{S}{N^2}$$

This is intuitive as full spillovers will imply that (almost) nothing is produced in period two. For reasons we have discussed before the profits will still be the cartel profits, i.e. the monopoly profit S divided by the number of firms N. Period three profit remains the same.

2.2 The Equilibrium amount of entry

Following the process of backward induction we will finally determine entry behaviour in stage one.

Proposition 4 *Given entry costs of $\varepsilon = 1$ and for all $\alpha, \beta \in [0,1]$ there exists a unique subgame perfect equilibrium of the overall game where the equilibrium number of active firms N^* is given implicitly by the* largest integer *number N that satisfies*

$$2\left\{1 - \frac{1}{\gamma}\frac{\alpha-\beta}{\alpha+\beta(N-1)}\left[N + \frac{1}{N} - 2\right]\right\} \geq \frac{N^2}{S} \tag{23}$$

Proof:

Entry will take place until the next firm will reduce costs below the sunk entry cost of $\varepsilon = 1$.[5] Hence the equilibrium number of firms is the largest integer N that satisfies

$$2\frac{S}{N^2}\left\{1 - \frac{1}{\gamma}\frac{\alpha-\beta}{\alpha+\beta(N-1)}\left[N + \frac{1}{N} - 2\right]\right\} \geq 1 \tag{24}$$

rearranging this equation yields the result. ¥

Lemma 2 The *equilibrium number of firms* N^* has the following comparative statics

$$\frac{\partial N^*}{\partial \beta} > 0, \frac{\partial N^*}{\partial \alpha} < 0, \frac{\partial N^*}{\partial \gamma}|_{\alpha>\beta} > 0, \frac{\partial N^*}{\partial \gamma}|_{\beta>\alpha} < 0.$$

The intuition is again straightforward and the effects are symmetric to those on profits: Larger inter-firm learning and lower intra-firm spillovers lead to larger output and lower equilibrium profits and therefore lower entry in stage one of the game. The effect of the overall elasticity of learning on the number of firms depends again on which effect dominates. If own firm learning is dominant, lower overall learning (higher γ) implies a lower individual output and by backward induction a larger number of entrants and vice versa.

[5] Note that having a sunk entry cost of ε (for example the initial cost to acquire a plant) is innocous here even though we will be talking about a bound for the equilibrium number of firms for the industry, since we will be looking at limit results and thus we can make ε arbitrary small relative to total industry demand.

2.3 Finding a lower bound to concentration

Theorem 1 If $\alpha > \beta(\gamma + 1)$ a lower bound to concentration exists. If $\alpha \leq \beta(\gamma + 1)$ the equilibrium number of firms N^* goes out of bounds as the size of the market S gets large, therefore a lower bound to market concentration fails to exist.

This finding may seem surprising at first sight given the opposite but *symmetric* effect that α and β have on the profit function. Requiring that intra-firm learning effects are simply stronger than inter-firm learning spillovers (i.e. $\alpha > \beta$) is not enough to guarantee a lower bound to concentration.

Looking at the optimal quantity choice (13) again and reminding ourselves that for equally strong effects ($\alpha = \beta$) or for no overall learning $\gamma \to \infty$ we will find the one shot homogeneous good Cournot quantity as the optimal output (for which no lower bound to concentration exists) we see that an *asymmetry* of the two effects is required for the existence of a bound.

For $\alpha < \beta$ a lower bound to concentration never exists (see Appendix). Individual learning effects are too weak relative to spillover effects for *any* degree of overall learning γ. Thus we require individual intra-firm learning to be stronger in order to get an overproduction result in stage two that lowers overall profits and bounds entry in stage one of the game. The asymmetry thus has to be in favour of intra-firm learning effects.

However simple dominance of the form $\alpha > \beta$ is only necessary, not sufficient for the existence of a bound. Individual intra-firm learning has to be supported by a strong overall learning effect in order to be able to induce the overproduction that limits entry and creates a lower bound to concentration, i.e. satisfies the condition $\alpha > \beta(\gamma + 1)$.

Hence despite the apparently symmetric effects of inter- and intra-firm learning on profits the existence of a lower bound involves as an additional requirement the degree of overall learning. It requires an asymmetry of the former effects in favour of inter-firm learning that needs to be larger the less pronounced the degree of overall learning is.

Solving the left hand side (LHS) of the equilibrium condition for N

$$2\left\{1-\frac{1}{\gamma}\frac{\alpha-\beta}{\alpha+\beta(N-1)}\left[N+\frac{1}{N}-2\right]\right\}\geq 0 \tag{25}$$

we define $\tilde{N}(\alpha,\beta,\gamma)$ as the largest integer that satisfies (25), i.e. gives an upper bound on the equilibrium number of firms in the industry. This implies a lower bound to market concentration in large markets. Even as the size of the market becomes unbounded there will be no entry of additional firms into the industry. Hence the $n-firm$ concentration ratio C_n remains bounded from zero.

Theorem 2 If $\tilde{N}(\alpha,\beta,\gamma)$ is well defined, then

$$\frac{\partial\tilde{N}(\alpha,\beta,\gamma)}{\partial\beta}>0,\frac{\partial\tilde{N}(\alpha,\beta,\gamma)}{\partial\alpha}<0,\frac{\partial\tilde{N}(\alpha,\beta,\gamma)}{\partial\gamma}>0. \tag{26}$$

We therefore find that the *upper bound to the equilibrium number of firms* in the industry $\tilde{N}(\alpha,\beta,\gamma)$ is decreasing in the level of intra-firm learning α and increasing in the level of inter-firm spillovers β and the overall learning elasticity parameter γ. The lower bound to concentration in large markets will move in the opposite direction.

Why is the upper bound on the equilibrium number of firms increasing in the overall learning elasticity parameter γ, i.e. why is it unambiguously larger for lower overall learning possibilities? This follows from noting that the condition for the bound to exist, $\alpha>\beta(\gamma+1)$ implies that $\alpha>\beta$ always holds for which the previous comparative statics show that equilibrium output in stage two is decreasing and total profits are increasing. Hence the equilibrium number of firms that enter in stage one and its upper bound (given it exists) will increase too.

2.3.1 Looking at the extreme spillover cases

Given that the number of firms $N > 1$ is now *continuous* and (23) holds with equality. When there are no inter-firm learning spillovers ($\beta = 0$) we find the derivative of LHS to be

$$\frac{\partial(LHS)}{\partial N}|_{\beta=0} = -\frac{2}{\gamma}\left(1 - \frac{1}{N^2}\right) < 0 \tag{27}$$

for any γ unless $\gamma \to \infty$. LHS will be a strictly decreasing function of N and it is not converging to some positive number as $\alpha > \beta(\gamma + 1)$ is always satisfied for any $\alpha \in [0, 1]$. Thus it will cut the $N - axis$ at some $N > 1$ where we find an upper bound to the equilibrium number of firms independent of market size. This is the same result as in Sutton (1998): Only in the complete absence of overall learning effects do we find no lower bound to market concentration.

When there are complete inter-firm learning spillovers ($\beta = 1$) we find

$$\frac{\partial(LHS)}{\partial N}|_{\beta=1} = 2(1-\alpha)\frac{N^2(1+\alpha) - 2N + 1 - \alpha}{\gamma(\alpha + N - 1)^2 N^2} \geq 0 \tag{28}$$

given $N > 1$ *for any* $\alpha \in [0, 1]$ and any γ. This implies that we do not find a lower bound to concentration for any degree of intra-firm learning. $\alpha > \beta(\gamma + 1) = (\gamma + 1)$ always fails to hold.

Corollary 1 There exists a critical level of inter-firm learning β^* such that for $\beta \geq \beta^*$ there is no lower bound to concentration.

This follows directly from Theorem 1 which implies that a lower bound will not exist for $\beta \geq \beta^* = \frac{1}{\gamma+1}\alpha$. This obviously does not imply that there is no bounded equilibrium number of firms N^* for a finite market size S.

Intuitively what happens is the following: For an increase in inter-firm learning spillovers β relative to intra-firm learning α at a given elasticity γ each firm will decreases its output level in stage two as spillovers which are beneficial to its competitors start to outweigh its own learning effects. The increase in profits which follows from this output restriction will attract a larger number of entrants in stage one. Given that β has risen above β^*, equilibrium profit levels are such that for a growing market size an upper bound on N fails to exist and as market grows strongly the equilibrium number of entering firms N^* goes out of bounds.

3 Conclusion

As in Sutton's (1998) model the upper bound on the equilibrium number of firms increases in the *overall learning elasticity parameter*. However the reasoning underlying the effect is more complex. It follows from the fact that the lower bound to concentration will only exist if intra-firm learning dominates inter-firm spillovers and this *asymmetry* has to be stronger the weaker the overall learning effect.

If this asymmetry is sufficiently pronounced, a higher elasticity, implying lower overall learning effects will unambiguously decrease individual quantities, increase profits and lead to more firms entering at some sunk cost hence pushing the lower bound to concentration downwards. Larger intra-firm learning will lead to a decrease in the upper bound to the level of entering firms and hence move the lower bound to concentration upwards in large markets. Inter-firm spillovers will work against this effect.

Even if inter-firm spillovers are dominated by intra-firm learning effects an upper bound on the number of firms may not exist if the overall learning elasticity parameter is large, i.e. if the overall learning effect is too weak to get sufficient overproduction which will limit equilibrium profits and entry. The equilibrium number of active firms grows unlimited as market size goes out of bounds. Whence the n firm concentration ratio C_n will be negligible and the lower bound to market concentration breaks.

In practice large inter-firm learning spillovers are often prevented by *patents* and firm secrecy making the non-existence of the bound less likely. However Theorem 1 still has an important *policy implication*. It suggests a trade-off between inter-firm learning spillovers and overall learning. Given an industry where we observe a high global concentration ratio and can be certain that a lower bound holds due to very strong intra-firm and overall learning effects (e.g. memory microchips).

Here an exogeneous increase in the overall industry learning dynamics for example by some new technology may allow for an increase in the level of inter-firm spillovers (e.g. a relaxation of patent laws) *without* leading to a change in market structure, i.e. without an effect on entry or incumbents' profits. Even as this industry grows very large (or if the new technology reduces setup costs drastically) the trade-off guarantees that the conditions for a bound still hold and its position is unchanged which will prevent a fragmented market structure.

4 Appendix

Proof of Proposition 1:

Derivation of equilibrium prices (8) and Cournot-Nash profits (9) in a differentiated product industry:

The profit of firm i with common marginal cost level c is given as

$$\pi_i = \lambda u_i x_i - c x_i \tag{29}$$

Using the finding above that goods with positive sales in equilibrium must have prices proportional to their qualities (2) we find λ from the total budget condition as

$$\lambda = \frac{S}{\left(\sum_j u_j x_j\right)} \quad j = 1, ...i, ...N.$$

The optimal quantity choice for firm i is found from its first order condition, i.e. by differentiating (29) with respect to quantity and setting it equal to zero

$$\frac{\partial \pi_i}{\partial x_i} = \lambda u_i + u_i x_i \frac{\partial \lambda}{\partial x_i} - c = 0 \tag{30}$$

Differentiation of λ w.r.t. some x_i yields

$$\frac{\partial \lambda}{\partial x_i} = -\frac{S}{\left(\sum_j u_j x_j\right)^2} \frac{\partial}{\partial x_i}\left(\sum_j u_j x_j\right) = -\frac{S u_i}{\left(\sum_j u_j x_j\right)^2} = -\frac{u_i}{S}\lambda^2$$

Substituting for $\frac{\partial \lambda}{\partial x_i}$ and rearranging we find optimal quantities for firm i to be

$$u_i x_i = \frac{S}{\lambda} - \frac{cS}{\lambda^2}\frac{1}{u_i} \tag{31}$$

and summation over all j firm's products yields

$$\sum_j u_j x_j = \frac{NS}{\lambda} - \frac{cS}{\lambda^2}\sum_j \frac{1}{u_j} \tag{32}$$

Using the total budget condition (3) again we can rewrite this as

$$\frac{S}{\lambda} = \frac{NS}{\lambda} - \frac{cS}{\lambda^2}\sum_j \frac{1}{u_j} \tag{33}$$

and find

$$\lambda = \frac{c}{N-1} \sum_j \frac{1}{u_j} \tag{34}$$

Substitution for λ into (31) yields optimal qualities as

$$x_i^* = \frac{S}{c} \frac{N-1}{u_i \sum_j \frac{1}{u_j}} \left\{ 1 - \frac{N-1}{u_i \sum_j \frac{1}{u_j}} \right\} \tag{35}$$

We now *solve for equilibrium prices* using $p_j = \lambda u_j \; \forall \; j = 1, ...i, ...N$ so that from (34) we find the price for good i as

$$p_i^* = \frac{c u_i}{N-1} \sum_j \frac{1}{u_j} \tag{36}$$

Alternatively we can write

$$p_i^* - c = \left\{ \frac{u_i}{N-1} \sum_j \frac{1}{u_j} - 1 \right\} c \tag{37}$$

Substituting (35) and (37) into the profit function (29) after some simplification we find equilibrium profits of

$$\pi_i^* = \left\{ 1 - \frac{N-1}{u_i \sum_{j=1}^{N} \frac{1}{u_j}} \right\}^2 S$$

which is precisely $\pi(u_i \, | \mathbf{u}_{-i})$ as given in (9) if we 'normalize' profits by defining $\pi(\cdot \, | \cdot) \equiv \frac{\pi_i^*}{S}$. ¥

Intermediate steps in the Proof of Proposition 2:

Derivation of (16) and (17). Taking the derivative of the stage three profit function (11) with respect to u_i, the *quality* level of firm i, given all other firms have quality level $\bar{u}$, i.e. $(u_{-i}) = \bar{u}$ yields

$$\frac{\partial S\pi(u_i\,|\mathbf{u}_{-i})}{\partial u_i} = \frac{2S\bar{u}\left(u_i(N-1)+\bar{u}(2-N)\right)(N-1)^2}{\left(\bar{u}+u_i(N-1)\right)^3} \tag{38}$$

Write (38) as an elasticity of perceived quality on profits given that all firms have quality level $\bar{u}$, i.e. $(u_i) = \bar{u}\ \forall\ i$ of the form

$$\frac{u_i}{\pi}\frac{\partial \pi}{\partial u_i}\,|_{(u_i)=\bar{u}} = 2\frac{(N-1)^2}{N} \tag{39}$$

Note that symmetrically we find that from the Cournot profit function (9)

$$\frac{u_j}{\pi}\frac{\partial \pi}{\partial u_j}\,|_{(u_i)=\bar{u}} = -2\frac{(N-1)}{N} \tag{40}$$

so that firm $j's$ quality with $j \neq i$ will have a negative effect on firm $i's$ profit π so that elasticity of perceived quality u_j on profits π_i is negative. For a derivation see below.

Derivation of (39) and (40), the elasticity of perceived rival quality on firm $i's$ profits: From the Cournot profit function (9) *(note that the sum over j includes i)* with N firms we find that the derivative of profits for firm i with respect to its own quality level is

$$\frac{\partial \pi}{\partial u_i} = 2\left(1-\frac{N-1}{u_i\sum_j \frac{1}{u_j}}\right)\left(\frac{1}{u_i^2(\sum_j \frac{1}{u_j})}\left(N-1-\frac{N-1}{u_i\sum_j\frac{1}{u_j}}\right)\right)$$

and the derivative of profits for firm i with respect to one other firm $j's$ quality level is

$$\frac{\partial \pi}{\partial u_j} = -2\left(1-\frac{N-1}{u_i\sum_j\frac{1}{u_j}}\right)\left(\frac{N-1}{u_i\left(\sum_j\frac{1}{u_j}\right)^2 u_j^2}\right)$$

Writing these in terms of elasticities we find the desired results.

Rewriting *first part* of the right hand side (RHS) of the equation (15) as

$$\frac{\partial \pi}{\partial u_i}\frac{\partial u_i}{\partial x_i} = \left(\frac{u_i}{\pi}\frac{\partial \pi}{\partial u_i}\right) \times \left(\frac{\pi}{u_i}\frac{\partial u_i}{\partial x_i}\right) \tag{41}$$

using (39) and the elasticity formula for u_i which gives the elasticity of perceived quality in stage three with respect to stage two output as

$$\frac{x_i}{u_i}\frac{\partial u_i}{\partial x_i} = \frac{\alpha}{\gamma}\frac{x_i}{(\alpha x_i + \boldsymbol{\beta}'\mathbf{x}_{-i})} \tag{42}$$

we find

$$\frac{\partial \pi}{\partial u_i}\frac{\partial u_i}{\partial x_i}\Big|_{(u_i)=\bar{u}} = 2\frac{(N-1)^2}{N} \times \frac{\alpha}{\gamma}\frac{x_i}{(\alpha x_i + \boldsymbol{\beta}'\mathbf{x}_{-i})}\frac{\pi}{x_i} \tag{43}$$

The continuity of the support of α and β guarantees that $\alpha = (\beta_i) = 0 \ \forall \ i$ is a zero probability event. We seek a symmetric SPNE in which firms set a common output level x in stage two. We can therefore set $x_i = (\mathbf{x}_{-i}) = x$ to simplify the above to

$$\frac{\partial \pi}{\partial u_i}\frac{\partial u_i}{\partial x_i}\Big|_{(u_i)=\bar{u}} = 2\frac{(N-1)^2}{N} \times \frac{\alpha}{\gamma}\frac{1}{\alpha + \mathbf{i}'\boldsymbol{\beta}}\frac{\pi}{x} \tag{44}$$

We make an assumption about *full symmetry of spillovers* as follows. The complete learning technology can be written as

$$\mathbf{u} = \max(1, (\mathbf{B}\mathbf{x}))^{\frac{1}{\gamma}} \tag{45}$$

where $\mathbf{B}$ is a fully symmetric $N \times N$ matrix with generic elements $(\beta_{ij}) = (\beta_{ji}) = \beta \ \forall \ i \neq j$ and $(\beta_{ij}) = \alpha \ \forall \ i = j$ on the diagonal and $\mathbf{u}$ and $\mathbf{x}$ are $N \times 1$ column vectors of quality and quantity. This formulation can then be broken into one of the form

$$\mathbf{u} = \max(1, (\alpha\mathbf{x} + \boldsymbol{\Xi}\mathbf{x}))^{\frac{1}{\gamma}} \tag{46}$$

where $\boldsymbol{\Xi}$ is now a $N \times N$ matrix with $(\beta_{ij}) = 0 \ \forall \ i = j$ on the diagonal and β everywhere else. This guarantees that we can write the symmetric quality formula to (10) for some firm j as

$$u_j = \max(1, (\alpha x_j + \boldsymbol{\xi}_j'\mathbf{x}))^{\frac{1}{\gamma}} \tag{47}$$

where $\boldsymbol{\xi}_j'$ is the $j'th$ row of matrix $\boldsymbol{\Xi}$ of the generic form $(\beta_{j,1}, \beta_{j,2}, ...\beta_{j,j-1}, 0, \beta_{j,j+1}, ...\beta_{j,N})$. We now delete the empty diagonal by denoting $\boldsymbol{\beta}_{-j}'$ as the $1 \times N - 1$ row vector without the $j'th$ element and $\mathbf{x}_{-j}$ as the $N - 1 \times 1$ column vector of quantities without the $j'th$ element so that we can write

$$u_j = \max(1, (\alpha x_j + \boldsymbol{\beta}_{-j}' \mathbf{x}_{-j}))^{\frac{1}{\gamma}} \tag{48}$$

as the symmetric counterpart to (10) for firm j. Note that vector $\mathbf{x}_{-j}$ contains element (x_i).

Rewriting the *second part* of the RHS as

$$[\nabla \pi(\mathbf{u}_{-i})]' \nabla \mathbf{u}_{-i}(x_i) = \left(\frac{1}{\pi} [\nabla \pi(\mathbf{u}_{-i})]' \mathbf{u}_{-i} \right) \times \left([\nabla \mathbf{u}_{-i}(x_i)]' \frac{1}{\mathbf{u}_{-i}} \pi \right) \tag{49}$$

we find that the elasticity equation of quantity x_i with respect to the quality of an average other firm (under spillover symmetry 'any' other firm) $-i$ becomes

$$\nabla \mathbf{u}_{-i}(x_i) \frac{1}{\mathbf{u}_{-i}'} x_i = \frac{\mathbf{i}' \boldsymbol{\beta}}{\gamma(N-1)} \frac{x_i}{(\alpha x_i + \boldsymbol{\beta}' \mathbf{x}_{-i})} \tag{50}$$

so that using (40) for the aggregate of all $(N - 1)$ other firm we find that

$$[\nabla \pi(\mathbf{u}_{-i})]' \nabla \mathbf{u}_{-i}(x_i) \left|_{(u_i)=\bar{u}} \right. = -2 \frac{(N-1)^2}{N} \times \frac{\mathbf{i}' \boldsymbol{\beta}}{\gamma(N-1)} \frac{x_i}{(\alpha x_i + \boldsymbol{\beta}' \mathbf{x}_{-i})} \frac{\pi}{x_i} \tag{51}$$

In a symmetric SPNE the above simplifies to

$$[\nabla \pi(\mathbf{u}_{-i})]' \nabla \mathbf{u}_{-i}(x_i) \left|_{(u_i)=\bar{u}} \right. = -2 \frac{(N-1)^2}{N} \times \frac{\mathbf{i}' \boldsymbol{\beta}}{\gamma(N-1)} \frac{1}{\alpha + \mathbf{i}' \boldsymbol{\beta}} \frac{\pi}{x} \tag{52}$$

¥

Proof of Lemma 1:

See that the optimal stage two quantity x^* falls strictly in the level of spillovers β for all $N > 1$ as

$$\frac{\partial x^*}{\partial \beta} = -2 \left[N + \frac{1}{N} - 2 \right] \frac{S}{N} \frac{\alpha}{\gamma \left(\alpha + \beta(N-1)\right)^2 c} < 0 \tag{53}$$

and increases in the level of intra-firm learning α as

$$\frac{\partial x^*}{\partial \alpha} = 2\left[N + \frac{1}{N} - 2\right]\frac{S}{N}\frac{\beta}{\gamma\left(\alpha + \beta(N-1)\right)^2 c} > 0 \tag{54}$$

It also increases if the overall learning effect becomes more important, as

$$\frac{\partial x^*}{\partial \gamma} = -2\left[N + \frac{1}{N} - 2\right]\frac{S}{N^2}\frac{\alpha - \beta}{\gamma^2\left(\alpha + \beta(N-1)\right) c} < 0 \tag{55}$$

given that $\alpha > \beta$ holds and has the opposite sign if $\alpha < \beta$. ¥

Proof of Proposition 4 for a continuous number of firms:
The slope of LHS (25) which is now *binding* is

$$\frac{\partial(LHS)}{\partial N} = -2\left(\alpha - \beta\right)\left(N-1\right)\frac{N(\alpha+\beta) + \alpha - \beta}{\gamma\left(\alpha + \beta(N-1)\right)^2 N^2} \tag{56}$$

Thus LHS always attains an extreme point at $N = 1$ where it takes the value 2. Given $N = 1 + \varepsilon$ it follows from (23) that $S > \frac{1}{2}$ so that the RHS takes the value $\frac{1}{S} < 2$, i.e. *below* the LHS. For $\alpha > \beta$ the LHS is strictly decreasing and will cut the strictly increasing parabola on the RHS at some finite equilibrium value N^* for any bounded value of S. Both functions are continuous so that *existence* of N^* follows from the intermediate value theorem and uniqueness from the strictness property of both sides.

For $\alpha \leq \beta$ and $N > 1$ the LHS is weakly increasing. The second derivative of the LHS is

$$\frac{\partial^2(LHS)}{\partial N^2} = 4\left(\alpha - \beta\right)\frac{(N^3 - 3N)(\alpha\beta + \beta^2) + 2\beta^2 + 2\alpha\beta - \alpha^2}{\gamma\left(\alpha + \beta N - \beta\right)^3 N^3} \tag{57}$$

Again *existence* follows from the continuity of the functions. To check uniqueness for the case $\alpha \leq \beta$ where both the LHS and the RHS slope upwards we see from the above equation that the LHS is always weakly *concave* i.e. $\frac{\partial^2(LHS)}{\partial N^2} \leq 0$ for $N > 1$. As the RHS parabola is always strictly convex this guarantees a unique intersection of the two curves and hence a *unique* equilibrium number of firms. ¥

Proof of Lemma 2:
Although explicit calculation of N^* is very involved the comparative statics follows from simple observation of LHS. ¥

Proof of Theorem 1: An intercept with the $N - axis$ (and thus an upper bound on the equilibrium number of firms, $\tilde{N}(\alpha,\beta,\gamma)$) will exist if the LHS (25) *converges* to some negative number in the limit as $N \to \infty$ (not N^*). The limit result is

$$\lim_{N\to\infty} 2\left\{1 - \frac{1}{\gamma}\frac{\alpha-\beta}{\alpha+\beta(N-1)}\left[N + \frac{1}{N} - 2\right]\right\} = 2\frac{\beta(\gamma+1)-\alpha}{\gamma\beta} \tag{58}$$

Hence for $\alpha \le \beta(\gamma+1)$ the LHS will converge to some non-negative number and no intercept exists, i.e. there will be no upper bound on N^*. For for $\alpha > \beta(\gamma+1)$ such an intercept exists. For this not to violate the non-negativity constraint on total profits (quantities are always positive here) we require that $\gamma_{\alpha>\beta} \ge \frac{(N-1)^2}{N}$. We can easily prove by example that $\gamma < \frac{(N-1)^2}{N}$ is not a necessary condition for $\alpha > \beta(\gamma+1)$ *in equilibrium*. ¥

Proof of Theorem 2:

The two roots that satisfy the binding LHS (25) for $\alpha > \beta(\gamma+1)$ are given by

$$\tilde{N}_1(\alpha,\beta,\gamma) = \frac{1}{2}\frac{(\alpha-\beta)(\gamma+2) - \sqrt{\gamma}\sqrt{((\alpha-\beta)(\gamma(\alpha-\beta)+4\alpha))}}{\alpha-\beta(1+\gamma)} \tag{59}$$

and

$$\tilde{N}_2(\alpha,\beta,\gamma) = \frac{1}{2}\frac{(\alpha-\beta)(\gamma+2) + \sqrt{\gamma}\sqrt{((\alpha-\beta)(\gamma(\alpha-\beta)+4\alpha))}}{\alpha-\beta(1+\gamma)} \tag{60}$$

Note that both roots will always be positive as $\alpha > \beta(\gamma+1) \Rightarrow \alpha > \beta$ so that the numerator and the denominator are always positive.

$\tilde{N}_1(\alpha,\beta,\gamma)$ however does not satisfy $N > 1$ for $\beta \in [0,1]$ and $\alpha > \beta(\gamma+1)$. Proof by contradiction:

$$\tilde{N}_1(\alpha,\beta,\gamma) > 1 \Leftrightarrow$$

$$\underbrace{\alpha - \beta - \frac{\sqrt{\gamma}}{\gamma}\sqrt{((\alpha-\beta)(\gamma(\alpha-\beta)+4\alpha))}}_{\Psi} > -2\beta \tag{61}$$

For $\alpha = \beta(\gamma + 1)$ the above equation (61) holds with equality. We now show that Ψ is *strictly decreasing* in α. The derivative of Ψ w.r.t α is

$$\frac{\partial \Psi}{\partial \alpha} = 1 - \frac{\gamma (\alpha - \beta) + 4\alpha + (\alpha - \beta) (\gamma + 4)}{2\sqrt{\gamma}\sqrt{((\alpha - \beta) (\gamma (\alpha - \beta) + 4\alpha))}} \tag{62}$$

thus we need to show that

$$\gamma (\alpha - \beta) + 4\alpha + (\alpha - \beta) (\gamma + 4) > 2\sqrt{\gamma}\sqrt{((\alpha - \beta) (\gamma (\alpha - \beta) + 4\alpha))}$$

or

$$\frac{\alpha(\gamma + 4) - \beta(\gamma + 2)}{\gamma} > \sqrt{(\alpha - \beta)^2 + \frac{4\alpha(\alpha + \beta)}{\gamma}}$$

Completion of the square yields

$$\frac{\alpha(\gamma + 4) - \beta(\gamma + 2)}{\gamma} + \frac{2\alpha}{\gamma} > \sqrt{\left((\alpha - \beta) + \frac{2\alpha}{\gamma}\right)^2}$$

from which we find

$$2\alpha - \beta > 0$$

This always holds as the condition $\alpha > \beta(\gamma + 1) \Rightarrow \alpha > \beta$.

Hence for some $\alpha' > \alpha = \beta(\gamma + 1)$ we find

$$\alpha' - \beta - \frac{\sqrt{\gamma}}{\gamma}\sqrt{((\alpha' - \beta) (\gamma(\alpha' - \beta) + 4\alpha))} < -2\beta \tag{63}$$

which is a contradiction to (61). We conclude that $0 < \tilde{N}_1(\alpha, \beta, \gamma) < 1$.

The comparative statics of the relevant root $\tilde{N}_2(\alpha, \beta, \gamma) \equiv \tilde{N}(\alpha, \beta, \gamma)$ can be seen most easily by investigating the binding LHS (25)

$$2\left\{1 - \frac{1}{\gamma}\frac{\alpha - \beta}{\alpha + \beta(N - 1)}\left[N + \frac{1}{N} - 2\right]\right\} = 0$$

and noting that given $N > 1$

$$\frac{\partial(LHS)}{\partial \alpha} < 0, \frac{\partial(LHS)}{\partial \beta} > 0, \frac{\partial(LHS)}{\partial \gamma}\,|_{\alpha>\beta} > 0, \frac{\partial(LHS)}{\partial \gamma}\,|_{\alpha<\beta} < 0$$

As $\tilde{N}(\alpha, \beta, \gamma)$ denotes the intercept of LHS (25) with the $N - axis$ given $\alpha > \beta(\gamma+1)$ and LHS is monotonous and non-increasing for $\alpha \geq \beta$, we know that $\tilde{N}_2(\alpha, \beta, \gamma)$ will change accordingly. ¥

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