Problem set 2:
Partial equilibrium stochastic growth model

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1 Partial equilibrium stochastic growth model

The consumer solves the following problem:

$$\max_{t=0}^{\infty} \sum_{t=0}^{\infty} \beta^t E u(c_t)$$

subject to a budget constraint

$$c_t = A_t k_t^\alpha - i_t$$

and borrowing constraint

$$k_t \geq 0$$

and law of motion for capital

$$k_{t+1} = (1 - \delta) k_t + i_t$$

where \(i\) is investment, \(A\) is a productivity shock, and \(\delta\) is the depreciation rate.

Productivity follows a 2-state Markov process:

$$\begin{bmatrix}
A_{low} & A_{low}' \\
A_{high} & A_{high}'
\end{bmatrix}
\begin{bmatrix}
\pi_{ll} & \pi_{lh} \\
\pi_{hl} & \pi_{hh}
\end{bmatrix}$$

1.1 Major characteristics

- contraction mapping: model converges regardless of initial choice for value function
- intertemporal problem
- infinite horizon: we iterate on value function until it "converges" (in practice, until differences between value functions and/or policy functions become "small")
1.2 Bellman formulation

- states: $k$, $A$
- controls: $c$, $k'$, where one implies the other through budget constraint (i.e. we need only one)

\[
V(k, A) = \max_{c,k'} \{ u(c) + \beta EV(k', A') \}
\]

\[
V(k, A) = \max_{k'} \{ u(Ak'^\alpha + (1-\delta)k - k') + \beta EV(k', A') \}
\]

1.3 Calibration

Parameter values (annual calibration): $\delta = 0.08$, $\alpha = 0.33$, $\beta = 0.95$, $A_{\text{low}} = 0.9$, $A_{\text{high}} = 1.1$, $\pi_{l'} = 0.3$, $\pi_{lh'} = 0.7$, $\pi_{hh'} = 0.7$, $\pi_{l'lh'} = 0.3$, $\pi_{l'hh'} = 0.7$

Utility function: $u(c_t) = \ln(c_t)$

1.4 What to do

1. Solve this program numerically (you can use a setup analogous to the life cycle problem):

   (a) set up a grid for capital:
   - one way to get at a reasonable grid is to start with a coarse grid with a large maximum capital stock, let the program run, see which actual maximum capital holdings you get, and then decrease the maximum capital stock and make the grid finer
   - another, more theoretically founded, way is to calculate the steady state capital stock in the deterministic model with $A = A_{\text{high}}$, and then take as the maximum capital stock e.g. twice the size of this capital stock; here you get (from $f'(k) - \delta = r$)

   \[
   \alpha Ak'^{-1} = r + \delta
   \]

   Setting $r = \frac{1-\beta}{\beta}$, we hence get

   \[
   k = \left( \frac{1-\beta}{\alpha A} + \delta \right)^{\frac{1}{1-\beta}}
   \]

   which under the current calibration amounts to $k = 4.496$

   (b) set up a value function and decision function with dimension (number of points in capital grid $\times 2$), i.e. the functions have one entry for every combination of any point on the capital grid and the high and low productivity state
(c) iterate on the value function until it "converges" (of course, you are free to use policy function iteration instead)

2. Plot the value functions, i.e. plot $V(k, A_{low})$ and $V(k, A_{high})$

3. Plot the policy functions, i.e. plot $k'(k, A_{low})$ and $k'(k, A_{high})$

4. I you can, try to calculate and plot the steady-state distribution of capital

- generate a vector with one row for every possible state (i.e. number of rows: $2 \times$ number of points in capital grid)
- assume uniform distribution of agents of mass 1 over this vector (i.e. in every row, there are $1/(2 \times nk)$ agents)
- calculate transition matrix between different states based on policy functions and transition probabilities between $A_{low}$ and $A_{high}$
- multiply and re-multiply the initial vector with transition matrix until distribution on vector "converges" (in practice, until difference becomes "small")