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Matching markets with adverse selection ☆

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Abstract

This paper considers a market with adverse selection in the spirit of Rothschild and Stiglitz (Quart. J. Econ. 90 (1976) 629). The major departure from existing approaches is that we model a decentralized market that is open-ended and constantly refilled by new participants, e.g., by new workers and firms in the case of a labor market. The major novelty of this approach is that the distribution of types in the market becomes an endogenous variable, which is jointly determined with equilibrium contracts. As frictions become small, we show that the least-cost separating contracts are always supported as an equilibrium outcome, regardless of the distribution of types among entrants. Moreover, we derive conditions under which this outcome is also unique.

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1. Introduction

In the standard model of adverse selection, equilibrium contracts are determined for an exogenously given distribution of types. The underlying picture is that of a

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market of fixed size that clears instantaneously. In many markets, however, new market participants arrive continuously, replacing those who have traded successfully. In such a setting, the composition of the market must be treated as an endogenous variable and must be determined jointly with equilibrium contracts.

To capture these features, this paper considers a decentralized market that is constantly refilled by new entrants. There are two groups of entrants: agents with private information about their type and principals. Agents and principals negotiate pairwise over a transfer and an additional variable that satisfies a standard sorting (or single-crossing) condition with respect to the agent's type. Negotiations are modeled by allowing either side to make an offer with positive probability. This setting could, for instance, capture a labor market where workers are privately informed about their ability. Specifying production quotas, overtime hours, or training requirements may be a way to separate between more and less able workers.[1]

Our model stand in sharp contrast to the standard model of adverse selection. There, the market clears in one period, in which a fixed number of principals competes by publicly making offers to a fixed number of agents. If an equilibrium in pure strategies exists, it is unique and separating, leading to the least-cost separating (or "Rothschild–Stiglitz") contracts. However, an equilibrium in pure strategies typically does not exist if the fraction of low types is sufficiently small, in which case high types would prefer to be pooled with low types.[2] In contrast, our model of a decentralized and ongoing market supports the Rothschild–Stiglitz contracts for *any* proportion of types entering the market. The key difference to the standard model is that the distribution of types in the market is endogenous and determined jointly with equilibrium contracts.

For an illustration of the key mechanism in this paper, take the case where the fraction of low types in each new cohort is small. In equilibrium, low types will circulate longer than high types. Consequently, the fraction of low types in the market exceeds that in each new cohort of entrants. Low types circulate longer as, in equilibrium, principals find it optimal to (also) offer contracts that are only acceptable to high types. As market frictions vanish, we can support the Rothschild–Stiglitz contracts as a limit outcome. What is more, we show that this is also the unique outcome if either of the following two conditions applies. First, it is sufficient that the Rothschild–Stiglitz contracts are interim efficient given the distribution of types among entrants. Second, the Rothschild–Stiglitz outcome is also unique if the probability with which an agent can make an offer is sufficiently small. Hence, the Rothschild–Stiglitz contracts are the unique outcome if we only deviate "marginally" from the standard model in that it is almost always principals who make offers.[3]

[1] For an application of adverse selection models to labour markets see, for instance, the case of law firms and working hours in [14].

[2] An alternative approach are signaling games in which agents propose contracts (see [13]). Signaling games support a wide range of equilibrium outcomes, including pooling contracts.

[3] I thank the referee for suggesting this interpretation.

This paper is not the first one that offers a "solution" to the existence problem of Rothschild and Stiglitz [20]. The strand of the literature closest to ours has explored variations of the basic non-cooperative (or strategic) approach. Wilson [23] proposes an "anticipatory" equilibrium concept and Riley [19] a "reactive" equilibrium concept, both of which expand the strategic options of principals. (See also [9].) Inderst and Wambach [11,12] introduce capacity constraints into the basic model. That is, firms may only have a fixed number of vacancies or insurance companies may be constrained by their committed capital. If total capacity in the industry is limited and sufficiently dispersed among firms, Inderst and Wambach show that the Rothschild–Stiglitz contracts can always be supported as the unique equilibrium outcome. The crucial difference is that, with limited capacity, a deviating offer will no longer attract a fair distribution of types as different types have different gains from the new contract and, therefore, a different willingness to accept rationing.[4] Other strands of the literature have considered non-cooperative (or axiomatic) concepts (e.g., [3])[5] or a general-equilibrium approach, where all participants act as price takers (e.g., [18]).[6]

This paper is also related to recent research studying private information in matching and search environments. Most of this literature assumes that a fraction of market participants is informed about the value of a traded commodity. In contrast, in our setting each agent knows only his own type, which represents an independent random draw. Most related are Moreno and Wooders [16] and Blouin [2], where agents are also "informationally small" and where the distribution of types is endogenous. They consider simple contracts over transfers without a sorting variable and are interested in non-stationary equilibria where goods of high quality are only sold at certain points of time. Instead, our focus is on the existence and characterization of stationary equilibria if contracts admit a sorting variable.

The rest of this paper is organized as follows. In Section 2 we introduce the model. Section 3 derives the main results and Section 4 concludes. All proofs are relegated to the appendix.

[4] The mechanism at work is thus quite different from the one in the current paper. Inderst and Wambach [11,12] follow the standard approach in that the distribution of types in the market is exogenous, principals publicly announce offers, and the market subsequently clears. Limited capacity matters only in that it allows to endogenize the distribution of types in the "queue" attracted by a deviating offer. In contrast, in the current paper a principal always faces the market distribution, which is, however, endogenous.

[5] For an overview of different approaches see also Gale [8]. Myerson [17] has proposed an axiomatic approach that is somewhat similar in spirit to ours. There, the basic concepts are "matching plans", which stipulate contracts and trading probabilities for participating types. The profitability of competing (or deviating) matching plans depends then crucially on the distribution of types in active matching plans.

[6] One strand essentially considers a market for each possible contract. Gale [7] and Dubey et al. [5] have introduced refinements to deal with the vast set of equilibria supported by the freedom to keep markets "inactive". Bisin and Gottardi [1] consider only linear price schedules. There, uniform prices are determined for each type of agent and each state.

2. The model

2.1. The economy

We consider a stationary market populated by principals and agents. Principals and agents can contract pairwise. A contract $c \in C$ prescribes a monetary transfer t and an additional real-valued variable x. It is convenient to restrict x to non-negative values. Agents can be of two types, denoted by $i \in I = \{1, 2\}$. An agent's type is his private information. The utility of an agent with type i is given by $V_i(c) := v_i(x) + t$ and that of a principal by $U_i(c) := u_i(x) - t$. Note that we consider common values, i.e., the principal's utility depends on the agent's type.[7] Utility functions are assumed to be continuously differentiable.[8] Denote $w_i(x) := u_i(x) + v_i(x)$. We invoke the following standard assumptions:

(A.1) $w_i(x)$ is strictly quasiconcave; $dw_i(x)/dx < 0$ holds for high values of x.
(A.2) $dv_2(x)/dx > dv_1(x)/dx$.
(A.3) $u_2(x) > u_1(x)$.

By (A.3) principals prefer to contract with high types, while by (A.2) the variable x satisfies a standard sorting condition. Our main application, to which we frequently return, is that of a labor market. In this application, agents are workers, principals are employers with a single vacancy, t denotes the wage, and x may represent a measure of output, working hours, or hours of required training. High types are more productive and incur less disutility from putting in an extra hour of work or training.

Principals and agents interact in an anonymous market with random matching. Time runs discretely and payoffs are discounted by $0 < \delta < 1$. Each period the mass one of agents enters the market, of which the fraction $0 < \mu^0 < 1$ consists of high types. To determine the number of entering principals we can adopt two alternative specifications. Following a standard approach in the labor literature, we can assume that the entry of new principals is determined by a zero-profit condition to open up new trading opportunities, i.e., vacancies. Each principal incurs the up-front costs $U^0 > 0$. Alternatively, we can assume that each period a new cohort of principals arrives at the market fringe. A new cohort of principals is larger than that of agents, i.e., their mass exceeds one. Instead of entering the market, principals can also choose an alternative option, which realizes $U^0 > 0$. As principals discount future payoffs and as the market is stationary, they will either enter the market immediately or take up their outside option.

Under either of the two depicted scenarios, principals will realize U^0 in the market. This specification, i.e., that principals in essence compete for agents, ensures that we can compare our results to those obtained in more standard approaches, where it is

[7] Inderst [10] also uses a matching market to study contracting under private values, i.e., where the agent's type does not directly enter the principal's utility. There, the non-existence problem of Rothschild and Stiglitz [20] does not arise and there is no role for an endogenous adjustment of the distribution of types.

[8] Differentiability is invoked to facilitate the proof of the convergence results.

typically assumed that principals represent the long side of the market and compete themselves down to zero profits.

To ensure that the market is stationary, the same mass of agents and principals must enter each period, that is the mass one. The stock of principals and agents in the market is determined endogenously. We denote the mass of principals by s^{P} and that of agents of type i by s_i^{A}. We further denote the distribution of types in the market by $\mu := s_2^{\mathrm{A}}/s^{\mathrm{A}}$, where $s_{\mathrm{A}} := s_1^{\mathrm{A}} + s_2^{\mathrm{A}}$, and the ratio of principals to agents by $\theta := s^{\mathrm{P}}/s^{\mathrm{A}}$. The ratio θ determines the matching probability of agents, which we denote by $m^{\mathrm{A}}(\theta)$, and the matching probability of principals, which we denote by $m^{\mathrm{P}}(\theta)$. Both probabilities are assumed to be continuous, while $m^{\mathrm{A}}(\theta)$ is increasing and $m^{\mathrm{P}}(\theta)$ decreasing. Furthermore, it holds that $m^{\mathrm{A}}(0) = 0$, $m^{\mathrm{P}}(0) = 1$, $\lim_{\theta \to \infty} m^{\mathrm{P}}(\theta) = 0$, and $\lim_{\theta \to \infty} m^{\mathrm{A}}(\theta) = 1$.[9] In what follows, we will frequently omit the variable θ in m^{A} and m^{P}.

A common choice is the "classical" matching technology, according to which agents are matched with probability $\min\{1, \theta\}$ and principals with probability $\min\{1, 1/\theta\}$. Alternatives are the "proportional" technology, where principals match with probability $s^{\mathrm{A}}/(s^{\mathrm{A}} + s^{\mathrm{P}})$ and agents with probability $s^{\mathrm{P}}/(s^{\mathrm{A}} + s^{\mathrm{P}})$, and the "exponential" technology, which specifies $m^{\mathrm{P}} = (1 - e^{-\theta})/\theta$ and $m^{\mathrm{A}} = 1 - e^{-\theta}$.

In a given match, the principal is chosen with probability $b \in (0, 1)$ to make an offer, while otherwise the agent is chosen. If the offer is rejected, the match breaks up. In both cases, i.e., if the principal makes an offer (screening) or if the agent makes an offer (signaling), the respective party can offer a menu of deterministic contracts.[10] Observe that we endow both parties with the right of proposal. By waiting sufficiently long, any market participant can be sure that he will be chosen as the proposer in some match. Allowing both sides of the market to become active is a crucial ingredient of any matching model. Otherwise, one encounters the well-known monopoly price paradox [4]. In particular, if only principals had the right of proposal, we would be back to solving the problem of a monopolistic principal.[11]

2.2. Equilibrium requirements

We will now be rather brief in specifying the equilibrium requirements, which are quite standard. As noted above, we require the market to be stationary. We also restrict consideration to equilibria where players choose symmetric, stationary, and sequentially optimal strategies in the contracting games. Moreover, in the signaling game principals update their beliefs according to Bayes' rule. We denote a principal's

[9] As $m^{\mathrm{A}}(\theta) = \theta m^{\mathrm{P}}(\theta)$, we also have that $m^{\mathrm{P}}(\theta) - m^{\mathrm{A}}(\theta)$ is strictly decreasing.

[10] Allowing for a menu also in the signaling game follows Maskin and Tirole [15].

[11] A random draw of the proposer amounts to a lottery over contracts. As contracts specify more than just a transfer and as the surplus function is strictly quasiconcave in the additional variable x, such a lottery may not be efficient from an ex-ante perspective. However, as bargaining over the right of proposal would have to proceed under private information, this inefficiency may not be resolved even in more flexible contractual games.

payoff in the screening game by U^{P} and his payoff in the signaling game by U^{A}. An agent's respective payoffs are denoted by V_i^{P} and V_i^{A}.

Suppose a match is broken up unsuccessfully and the two sides search anew. We denote the respective (reservation) payoff for a principal by U^{R} and that for an agent by V_i^{R}. From stationarity we obtain

$$U^{\mathrm{R}} = \delta[(1-m^{\mathrm{P}})U^{\mathrm{R}} + m^{\mathrm{P}}(bU^{\mathrm{P}} + (1-b)U^{\mathrm{A}})],$$
$$V_i^{\mathrm{R}} = \delta[m^{\mathrm{A}}V_i^{\mathrm{R}} + (1-m^{\mathrm{A}})(bV_i^{\mathrm{P}} + (1-b)V_i^{\mathrm{A}})]. \tag{1}$$

It is now convenient to assume that, after entry, it takes one period to start searching for a trading partner. This assumption is without consequences for our results as we will focus below on high values of δ. It implies that the payoff from entering the market is equal to the reservation value, i.e., to U^{R} for principals and to V_i^{R} for agents. As an immediate implication, we have from stationarity that $U^{\mathrm{R}} = U^0$.

We denote by $\pi_i(c)$, where $c \in C$, the equilibrium distribution of contracts for type i. This distribution is constructed as follows. Note first that matches can be dissolved unsuccessfully. Denoting the case of break-up by $\emptyset$, let $\alpha_i^{\mathrm{A}}(\emptyset)$ be the probability with which the signaling game with type i is dissolved unsuccessfully. In the screening game, type i is unsuccessful with probability $\alpha_i^{\mathrm{P}}(\emptyset)$. In general, we denote the outcome of the signaling game by a distribution $\alpha_i^{\mathrm{A}}(c)$ over $c \in C^0 = C \cup \{\emptyset\}$ and that of the screening game by a distribution $\alpha_i^{\mathrm{P}}(c)$ over $c \in C^0$. We next denote the aggregate probability with which type i will break up a match by

$$\xi_i := b\alpha_i^{\mathrm{P}}(\emptyset) + (1-b)\alpha_i^{\mathrm{A}}(\emptyset).$$

Using stationarity, the distribution over implemented contracts $c \in C$ becomes then

$$\pi_i(c) := \frac{b\alpha_i^{\mathrm{P}}(c) + (1-b)\alpha_i^{\mathrm{A}}(c)}{1-\xi_i}. \tag{2}$$

Note that, for the market to be stationary, the probability of failure ξ_i must not be equal to one for either type. Below we make an additional assumption that ensures that contracting is indeed profitable for both types. We refer to the pair of distributions over implemented contracts $\pi := \{\pi_i\}_{i \in I}$ as an allocation.

For future reference we finally provide a formula for the distribution of agents in the market. The distribution depends on the agents' success in the screening and signaling games. Recall that each period the mass μ^0 of high-type agents and the mass $1-\mu^0$ of low-type agents enter the market. A given agent of type i is successful with probability $1-\xi_i$. Multiplying this probability with the mass of agents of type i, i.e., with s_i^{A}, we obtain the mass of exits, which by stationarity must be equal to that of entrants of type i. This yields the requirements $s_1^{\mathrm{A}}(1-\xi_1) = 1-\mu^0$ and $s_2^{\mathrm{A}}(1-\xi_2) = \mu^0$, from which we obtain the distribution

$$\mu := \frac{\mu^0(1-\xi_1)}{\mu^0(1-\xi_1) + (1-\mu^0)(1-\xi_2)}. \tag{3}$$

3. Analysis

3.1. Preliminary remarks

As noted in the Introduction, the most prominent approach to markets with adverse selection is to consider a one-shot game of screening. If there exists an equilibrium in pure strategies, the outcome is uniquely determined. We call the respective contracts the Rothschild–Stiglitz (RS) contracts and denote them by $\{c_i^{\mathrm{RS}}\}_{i\in I}$. They solve the following program.

Program for the RS contracts: For $i=1$ the contract c_1^{RS} maximizes $V_1(c)$ subject to $c\in C$ and $U_1(c)\geqslant U^0$. For $i=2$, the contract c_2^{RS} maximizes $V_2(c)$ subject to $c\in C$, $U_2(c)\geqslant U^0$, and $V_1(c)\leqslant V_1(c_1^{\mathrm{RS}})$.

This program ensures that separation for $i=2$ is achieved at least costs. Denote the realized utilities by $V_i^{\mathrm{RS}}:=V_i(c_i^{\mathrm{RS}})$. The following result is standard given Assumptions (A.1)–(A.3).

Lemma 1. *The family $\{c_i^{\mathrm{RS}}\}_{i\in I}$ is uniquely determined and satisfies $U_i(c_i^{\mathrm{RS}})=U^0$, $x_2^{\mathrm{RS}}\geqslant x_1^{\mathrm{RS}}$, and global incentive compatibility (i.e., $V_i^{\mathrm{RS}}\geqslant V_i(i,c_j^{\mathrm{RS}})$ for all $i,j\in I$).*

Define next the RS allocation $\pi^{\mathrm{RS}}:=\{\pi_i^{\mathrm{RS}}\}_{i\in I}$, where the distribution π_i^{RS} puts probability one on c_i^{RS}. We also make the following assumption, which ensures that contracting is profitable for both types.

Assumption (A.4): $V_i^{\mathrm{RS}}>0$ for all $i\in I$.

Consider once again the standard screening game. An equilibrium in pure strategies fails to exist if the RS allocation is not interim efficient given the prevailing distribution of types. Using the transferability of utility, we can give the following definition of interim efficiency.

Definition of interim efficiency: If $\tilde{\mu}$ represents the fraction of high types, the family of RS contracts $\{c_i^{\mathrm{RS}}\}_{i\in I}$ is said to be interim efficient, given $\tilde{\mu}$, if there does not exist a pair $(c_1,c_2)\in C^2$ such that $V_i(c_i)\geqslant V_i(c_j)$ for $i,j\in I$, $V_i(c_i)\geqslant V_i^{\mathrm{RS}}$ for $i\in I$, and $\tilde{\mu}U_2(c_2)+(1-\tilde{\mu})U_1(c_1)>U^0$.

Note that the RS contracts are always interim efficient regardless of the distribution of types if they are first-best, i.e., if c_2^{RS} also solves the relaxed problem to maximize $V_2(c)$ subject to $U_2(c)\geqslant U^0$. If c_2^{RS} differs from the first-best contract due to the binding incentive compatibility constraint for $i=1$, the RS family of contracts is only interim efficient if the high types' probability is sufficiently low. Otherwise, it is efficient to cross-subsidize the low type to ensure incentive compatibility at lower costs.

In what follows, we are mainly interested in the case where the discount factor δ approaches one. For this purpose, we define next the set of limit allocations $\bar{\Pi}$.

Definition of the set of limit allocations: An allocation π satisfies $\pi\in\bar{\Pi}$ if there exists a sequence of equilibria along which the discount factor δ converges to one and along which the respective equilibrium allocations, i.e., the pair of distributions over implemented contracts, converge (weakly) to π.

Before proceeding with the analysis, it is helpful to illustrate some of the definitions by means of a simple example.

Linear example: In the spirit of Spence [22], consider a labor contract where x denotes the amount of contractually specified training. Training serves only as a sorting device. That is, while being costly for workers it is completely unproductive. We specify $v_i(x) = -x/a_i$ and $u_i(x) = a_i$, where $a_2 > a_1 > U^0$. The RS contract for the low type specifies $x_1^{RS} = 0$ and $t_1^{RS} = a_1 - U^0$. For the high type we have $x_2^{RS} = a_1(a_2 - a_1)$ and $t_2^{RS} = a_2 - U^0$, which is only interim efficient if the probability of drawing a high type is sufficiently low and does not exceed $a_2/(a_1 + a_2)$.[12] Finally, we obtain $V_1^{RS} = a_1 - U^0$ and $V_2^{RS} = a_2 - U^0 - (a_2 - a_1)a_1/a_2$.

3.2. Existence of the RS allocation

The following is the key result of this paper.

Proposition 1. *The matching market supports the RS allocation, i.e.,* $\pi^{RS} \in \bar{\Pi}$.

Proof. See Appendix B.

Note first that the RS allocation is only a limit outcome. That is, we consider a sequence of equilibria where $\delta \to 1$. Clearly, as long as market participants are impatient, principals will exploit their contractual power in a screening game to extract more than U^0. The proof of Proposition 1 is only partially constructive and builds on a fixed point argument. To provide some intuition, we proceed as follows. We first offer some insight in the general logic of Proposition 1. Subsequently, we return to our linear example for an illustration.

Proposition 1 relies on the fact that the distribution of types is endogenous. Precisely, if the RS allocation is not interim efficient given μ^0, matches with low types must sometimes be broken up unsuccessfully, resulting in longer circulation for low types. Breaking up matches with low types is, in turn, only optimal for principals if the low types' reservation value is sufficiently high. This again holds only if low types can expect to obtain, at least with some probability, a cross-subsidized offer in the future. Hence, along the sequence of equilibria supporting π^{RS} in the limit we have (i) that low types are sometimes cross-subsidized and (ii) that their matches are sometimes broken up unsuccessfully. As $\delta \to 1$, the "amount" (or probability) of cross-subsidization will, however, go to zero.

[12] Formally, given the linearity in x one can restrict attention to either pooling contracts or separating menus without cross-subsidization. Unless $\mu \leqslant a_2/(a_1 + a_2)$, the principal's payoff under pooling, $\mu a_2 + (1 - \mu)a_1 - V_2^{RS}$, strictly exceeds that under separation, $\mu[a_2 - V_2^{RS} - (V_2^{RS} - V_1^{RS})a_1/a_2] + (1 - \mu)(a_1 - V_1^{RS})$.

The key mechanism in Proposition 1 is the endogeneity of the distribution of types in the market. Suppose that we took instead the composition of the market as a primitive, while choosing the flow of entrants to keep the market stationary. If μ is sufficiently high, one can show that the RS allocation cannot be supported as $\delta \to 1$.

The question which primitives to choose in "open-ended" (matching and search) markets has been addressed, in particular, by Gale [6]. An alternative approach followed, for instance, in [21] is to take the ratio of sellers to buyers in the market as exogenous. Even as $\delta \to 1$ the later approach leads to non-Walrasian outcomes. That is, irrespective of the overhang of sellers in the market, sellers capture a share of the surplus even in the frictionless limit $\delta \to 1$. Gale [6] suggested to take, instead, the flow of potential entrants as primitives, in which case the adjustment of market conditions ensures that the shorter side in terms of entrants—i.e., agents in our model—would obtain all surplus as $\delta \to 1$. By extending the analysis to markets with adverse selection, this paper extends the insights of Gale [6]. We show that it is also the distribution of types on a given side of the market, i.e., agents in our setting, that should be left to adjust endogenously.

Linear example: Suppose the RS contracts are not interim efficient as $\mu^0 < a_2/(a_1 + a_2)$. Denote this threshold by $\mu^{\mathrm{IE}} := a_2/(a_1 + a_2)$ such that $\mu^0 < \mu^{\mathrm{IE}}$. Suppose that each type of the agent offers his respective RS contract in the signaling game. We now focus on what happens in the screening game. Denote by ρ_1 the probability with which principals offer a single contract that is only acceptable to high types, leading to a break-up with low types. To make it optimal for principals not to offer an acceptable contract to low types, it must hold by construction of the RS contracts that $V_1^{\mathrm{R}} \geqslant V_1^{\mathrm{RS}}$. (Recall that $U^{\mathrm{R}} = U^0 = U_1(c_1^{\mathrm{RS}})$.) We construct a sequence of equilibria where $V_1^{\mathrm{R}} = V_1^{\mathrm{RS}}$ holds with equality. Given $\delta < 1$ and $V_1^{\mathrm{A}} = V_1^{\mathrm{RS}}$ in the signaling game, $V_1^{\mathrm{R}} = V_1^{\mathrm{RS}}$ requires that low types are with some probability cross-subsidized in the screening games. Denote by ρ_2 the probability with which principals offer a single pooling contract with $x = 0$ and $t = V_2^{\mathrm{R}} > V_1^{\mathrm{R}}$. Finally, with the residual probability $1 - \rho_1 - \rho_2$ principals offer a separating menu with $t_1 = V_1^{\mathrm{R}}$, $x_1 = 0$ and $x_2 = a_1(V_2^{\mathrm{R}} - V_1^{\mathrm{R}})$, $t_2 = V_2^{\mathrm{R}} - v_2(x_2)$. Note that x_2 achieves separation at least costs: The disutility for the low type, x_2/a_1, would exactly compensate for the difference in transfers, $t_2 - t_1 = a_2 - a_1$.

The key task is now to adequately choose ρ_1, the break-up probability, and ρ_2, the pooling probability, so as to support these strategies. When is a principal indifferent between offering separating and pooling contracts? The linear structure of our example allows for a simple answer: Regardless of the values V_i^{R}, a principal is exactly indifferent if $\mu = \mu^{\mathrm{IE}}$. (Compare also footnote 12.) By (3), $\mu = \mu^{\mathrm{IE}}$ holds if $\rho_1 = \frac{\mu^0 - \mu^{\mathrm{IE}}}{b\mu^0(1-\mu^{\mathrm{IE}})}$.[13] Note that ρ_1 is independent of δ. Next, the probability of pooling ρ_2 is pinned down by the requirement $V_1^{\mathrm{R}} = V_1^{\mathrm{RS}}$. We can show that $\rho_2 \to 0$ as $\delta \to 1$. As

[13] Note that our construction is only applicable if $\mu^0 - \mu^{\mathrm{IE}} < b\mu^0[1 - \mu^{\mathrm{IE}}]$, i.e., if $\mu^0 < \mu^{\mathrm{IE}}/(1 - b + b\mu^{\mathrm{IE}})$. For higher values of μ^0 we can, however, apply the same arguments by specifying, in addition, break-up in the signaling game.

also $V_i^{\mathrm{R}} \to V_i^{\mathrm{RS}}$, only RS contracts are therefore implemented in the limit, both in the signaling and in the screening game and both if principals offer an acceptable menu or only a contract to the high type. This is formally derived in Appendix A.

3.3. Characterization of the equilibrium set

From Proposition 1 we know that the RS allocation can always be supported as a limit outcome. In what follows, we provide a further characterization of the limit set. We proceed in two steps. First, we obtain an upper boundary that applies regardless of the chosen parameters. Second, we derive conditions under which the limit set is singular.

Consider for a moment a one-shot signaling game in which an agent offers a menu to a principal. The principal's prior beliefs are given by μ^0 and his reservation value equals U^0. This game has been analyzed in [15]. We find that the set of allocations supported by equilibria in this signaling game forms an upper boundary for our limit set $\bar{\Pi}$.

Proposition 2. *Any limit allocation $\pi \in \bar{\Pi}$ can also be supported by an equilibrium of the one-shot signaling game where the agent offers a menu and the principal has prior beliefs μ^0.*

Proof. See Appendix B.

We next provide two conditions under which the RS allocation is also the unique element of $\bar{\Pi}$. The first condition makes use of the screening game. We obtain uniqueness if the probability with which the principal is chosen to make an offer is sufficiently high. By the second condition, it is also sufficient that the RS contracts are interim efficient given the distribution among entrants.

Proposition 3. (*i*) *There exists a threshold $\bar{b} < 1$ for the probability with which principals make an offer such that for all $b > \bar{b}$ the RS allocation is the unique limit allocation, i.e., $\bar{\Pi} = \{\pi^{\mathrm{RS}}\}$.*

(*ii*) *If the RS allocation is interim efficient given the distribution among entrants μ^0, then it is the unique limit allocation, i.e., $\bar{\Pi} = \{\pi^{\mathrm{RS}}\}$.*

Proof. See Appendix B.

Consider first assertion (ii). We know from Maskin and Tirole [15] that type *i* can realize at least V_i^{RS} in the signaling game. As $\delta \to 1$ it is thus intuitive that V_i^{RS} represents a lower boundary for the agents' reservation value V_i^{R}. If the RS contracts are now interim efficient given μ^0, we can also show that V_i^{R} must not exceed V_i^{RS} as $\delta \to 1$. Intuitively, as all implemented contracts must satisfy the principals' individual rationality constraint as well as the incentive compatibility constraints of both types of the agent, there is no scope for either type to extract a higher payoff than V_i^{RS}. Once we have established that reservation values converge to the respective RS payoffs, it is straightforward that the same applies to the supported contractual set.

If the distribution among entrants is not interim efficient, assertion (i) shows another route to establish uniqueness. The limit set is always singular if the probability of playing the screening game b is sufficiently high. This finding is reassuring in the following sense. A matching model where the distribution of types is endogenously determined leads to a resolution of the existence problem (Proposition 1) while also selecting a unique outcome (Proposition 3) if we do not deviate "too much" from the standard screening approach; that is, if the probability of playing the screening game b is sufficiently high and, consequently, the probability of playing the signaling game $1 - b$ sufficiently low.

The intuition for assertion (i) is as follows. If low types were (non-marginally) cross-subsidized for high δ, we could show that it would be strictly optimal for principals to offer only high types an acceptable contract. As a consequence, low types could only trade if it was their turn to make a proposal. As b increases this becomes less likely. Consequently, low types circulate much longer and the distribution in the market deteriorates. But as μ becomes sufficiently low we can also rule out any (non-marginal) cross-subsidization in the signaling game.

4. Concluding remarks

This paper explores a new approach to analyze markets with adverse selection. We consider a stationary market with pairwise matching. In a given match, either side may have the right to propose a contract. In contrast to standard models of adverse selection, the dynamic (though stationary) nature of the considered market environment implies that equilibrium contracts and the distribution of types in the market are both endogenous and must, therefore, be jointly determined. We show that this allows to always support the Rothschild–Stiglitz contracts as a limit outcome, where the limit is taken by making market participants more and more patient. The key mechanism is that, in equilibrium, low types are not always offered an acceptable contract and thus circulate longer, implying that the market distribution deteriorates.

We also derive conditions under which the Rothschild–Stiglitz contracts are uniquely supported. This is the case if either principals are sufficiently likely to make an offer or if the Rothschild–Stiglitz contracts are interim efficient given the distribution among entrants. Obtaining the Rothschild–Stiglitz contracts as the unique outcome of a matching market with an endogenous distribution of types may lend further support to this prominent selection.

One of the restrictions in the paper is that we only consider the case of two types. While we conjecture that our results extend to larger type sets, there is no immediate way to extend some of our proofs. Also, the paper is focused on the limit outcome, i.e., on what happens as market participants become increasingly patient. Solving for equilibria with still substantial market frictions would be interesting. We conjecture that the degree to which the market distribution differs from that among entrants should be a function of market frictions and, thereby, of competition among principals. Possibly, such comparative results could allow to finally test whether the identified mechanism has some real-world relevance.

Appendix A. Linear example

To fully characterize a sequence of equilibria supporting the RS allocation in the limit we choose the classical matching technology. Focusing on sufficiently high values of δ and b, we can construct equilibria where more principals than agents are in the market, implying that $m^{\mathrm{A}} = 1$ and $m^{\mathrm{P}} = m = s^{\mathrm{A}}/s^{\mathrm{P}}$.

We specified that agents offer the RS contracts while principals randomize over the following offers: with probability ρ_2 they offer a pooling contract, where $x = 0$ and $t = V_2^{\mathrm{R}}$; with probability $1 - \rho_1 - \rho_2$ they offer a separating menu (c_1, c_2), where $t_1 = V_1^{\mathrm{R}}$, $x_1 = 0$, $x_2 = a_1(V_2^{\mathrm{R}} - V_1^{\mathrm{R}})$, and $t_2 = V_2^{\mathrm{R}} - v_2(x_2)$; and with probability ρ_1 they offer only c_2, which is acceptable only to high types. We already specified $\rho_1 = \frac{\mu^0 - \mu^{\mathrm{IE}}}{b\mu^0(1-\mu^{\mathrm{IE}})}$ to ensure that $\mu = \mu^{\mathrm{IE}}$. Next, we choose ρ_2 such that $V_1^{\mathrm{R}} = V_1^{\mathrm{RS}}$. For this note first that the low type realizes $V_1^{\mathrm{A}} = V_1^{\mathrm{RS}}$ and $V_1^{\mathrm{P}} = \rho_2 V_2^{\mathrm{R}} + (1 - \rho_2) V_1^{\mathrm{RS}}$, given that $V_1^{\mathrm{R}} = V_1^{\mathrm{RS}}$, while the high type realizes $V_2^{\mathrm{A}} = V_2^{\mathrm{RS}}$ and $V_2^{\mathrm{P}} = V_2^{\mathrm{R}}$. From (1) this yields $V_2^{\mathrm{R}} = \delta(1-b)V_2^{\mathrm{RS}}/(1-\delta b)$ and $V_1^{\mathrm{R}} = V_1^{\mathrm{RS}} = \delta b \rho_2 V_2^{\mathrm{R}}/[1-\delta(1-b\rho_2)]$, where we used that $m^{\mathrm{A}} = 1$. Substituting and solving for ρ_2, $V_1^{\mathrm{R}} = V_1^{\mathrm{RS}}$ holds if

$$\rho_2 = \frac{V_1^{\mathrm{RS}}}{V_2^{\mathrm{RS}}} \frac{1-\delta}{\delta b} \frac{1-\delta b}{\left[\delta(1-b) - \frac{V_1^{\mathrm{RS}}}{V_2^{\mathrm{RS}}}(1-\delta b)\right]}, \tag{A.1}$$

where $\rho_2 \to 0$ as $\delta \to 1$. This together with $V_2^{\mathrm{R}} \to V_2^{\mathrm{RS}}$ confirms convergence to the RS allocation as $\delta \to 1$. What remains to be shown is that we can specify $m = s^{\mathrm{A}}/s^{\mathrm{P}} < 1$ such that $U^{\mathrm{R}} = U^0$, given the specified strategies. By $U^{\mathrm{A}} = U^0$ and (1), $U^{\mathrm{R}} = U^0$ holds if

$$m = \frac{(1-\delta)U^0}{\delta b(U^{\mathrm{P}} - U^0)}. \tag{A.2}$$

By construction principals are indifferent between their three strategies while by $\mu = \mu^{\mathrm{IE}}$ we have $U^{\mathrm{P}} = U^0$ in case $V_2^{\mathrm{R}} = V_2^{\mathrm{RS}}$. Hence, we obtain $U^{\mathrm{P}} - U^0 = V_2^{\mathrm{RS}} - V_2^{\mathrm{R}}$, where substitution of V_2^{R} yields $V_2^{\mathrm{RS}} - V_2^{\mathrm{R}} = V_2^{\mathrm{RS}} \frac{1-\delta}{1-\delta b}$. Substitution into (A.2) yields finally $m = \frac{U^0}{V_2^{\mathrm{RS}}} \frac{1-\delta b}{\delta b}$, which indeed satisfies $0 < m < 1$ in case δ and b are sufficiently close to one.

Appendix B. Proofs

In what follows, we abbreviate some of the more tedious but intuitive steps of the proofs. Details can be found in the working paper version. We first present several auxiliary results. It also turns out to be more convenient to prove first Propositions 2 and 3 before turning to existence in Proposition 1.

B.1. Auxiliary results

Consider the screening game. Denote by B_i^{P} the intersection of C with the support of α_i^{P}. (Recall that the support of α_i^{P} can contain the null-contract $\emptyset$, which denotes break-up.) For the signaling game, denote the support in C by B_i^{A} and define $B_i = B_i^{\mathrm{P}} \cup B_i^{\mathrm{A}}$. Moreover, define

$$f = \frac{\delta m^{\mathrm{P}} b}{1-\delta[1-bm^{\mathrm{P}}]},$$
$$g = \frac{\delta m^{\mathrm{A}}(1-b)}{1-\delta[1-m^{\mathrm{A}}(1-b)]}, \tag{B.1}$$

where it follows from (1) that $V_i^{\mathrm{R}} = gV_i^{\mathrm{A}}$ in case $V_i^{\mathrm{P}} = V_i^{\mathrm{R}}$ and that $U^{\mathrm{R}} = fU^{\mathrm{P}}$ in case $U^{\mathrm{A}} = U^{\mathrm{R}}$. Lemma B.1 follows now from Maskin and Tirole [15].

Lemma B.1. $V_i^{\mathrm{A}} \geqslant \max\{V_i^{\mathrm{RS}}, V_i^{\mathrm{R}}\}$.

It is next intuitive that, in equilibrium, principals offer at least one type always an acceptable contract.

Lemma B.2. *If $\alpha_i^{\mathrm{P}}(\emptyset) > 0$ holds for some $i \in I$, $\alpha_j^{\mathrm{P}}(\emptyset) = 0$ holds for $j \neq i$.*

Proof. Omitted.

We can show next that the market distribution μ must remain bounded away from the corners zero and one. By Lemma B.2, this also implies that for each type the probability of break-up remains bounded away from zero.

Lemma B.3. *There exists a value $\Delta > 0$ such that, along any sequence of equilibria where $\delta \to 1$, $\xi_i(\delta) \to \bar{\xi}_i$, and $\mu(\delta) \to \bar{\mu}$, it holds that $\bar{\xi}_i < 1 - \Delta$ and $\Delta < \bar{\mu} < 1 - \Delta$.*

Proof. We need the following auxiliary result.[14]

[14] Note that we index a sequence of equilibria by way of writing, for instance, $V_i^{\mathrm{R}}(\delta)$ or $f(\delta)$.

Claim B.1. *Consider the set of incentive compatible contracts* $(c_1, c_2) \in C^2$ *satisfying* $V_2(c_2) \geqslant V_2^{RS}$, $V_1(c_1) > V_1^{RS}$, *and* $\mu U_2(c_2) + (1-\mu)U_1(c_1) \geqslant U^0$. *Then there exists* $\bar{\mu} > 0$ *such that this set is empty if* $\mu < \bar{\mu}$.

Proof. Consider the following program denoted by Γ: Choose $(c_1, c_2) \in C^2$ to maximize $V_2(c_2)$ subject to the constraints $V_1(c_1) \geqslant V_1^{RS}$, $V_1(c_1) \geqslant V_1(c_2)$, and $\mu U_2(c_2) + (1-\mu)U_1(c_1) \geqslant U^0$. By optimality, we have (i) that $x_1 = x_1^{RS}$, (ii) that the principal's constraint binds, and (iii) that at least one of the two constraints $V_1(c_1) \geqslant V_1^{RS}$ and $V_1(c_1) \geqslant V_1(c_2)$ binds. Moreover, note that the RS contracts are feasible and that, in case $V_1(c_1) = V_1^{RS}$, it is optimal to choose $c_2 = c_2^{RS}$. Hence, to prove Claim B.1 it is sufficient to show that, for all sufficiently low μ, $V_1(c_1) > V_1^{RS}$ implies $V_2(c_2) < V_2^{RS}$. We argue to contradiction and suppose that $V_1(c_1) > V_1^{RS}$, which by optimality implies $V_1(c_1) = V_1(c_2)$. Substituting from the binding constraints, it remains to maximize $\mu w_2(x_2) + (1-\mu)[v_2(x_2) - v_1(x_2)] - U^0$, which by (A.2) implies for $\mu \to 0$ that $x_2 \to \infty$. But by (A.1) this indeed implies for all sufficiently low μ that $V_2(c_2) < V_2^{RS}$. $\square$

For low μ, it follows intuitively from Claim B.1 and optimality for principals that there will be no cross-subsidization in the screening game. For the signaling game, a similar result follows by Claim B.1 and as $V_2^A \geqslant V_2^{RS}$ holds by Lemma B.1.

Claim B.2. *There exists* $\bar{\mu} > 0$ *such that:*[15]

(i) *If* (c_1, c_2) *with* $V_1(c_1) > V_1^{RS}$ *is accepted with positive probability in the signaling game, then the principal's posterior beliefs must put less than probability* $\bar{\mu}$ *on the low type.*

(ii) *If* $\mu < \bar{\mu}$ *then* $V_1^P = V_1^R$.

Proof. Omitted.

Suppose now there exists a sequence of equilibria where $\delta \to 1$ and $\mu(\delta) \to 0$. By Claim B.2, this implies $V_1^P(\delta) = V_1^R(\delta)$ for high δ such that $V_1^R(\delta) = g(\delta)V_1^A(\delta)$. Observe also that $\mu(\delta) \to 0$ implies $\alpha_1^A(\delta) \to 1$, i.e., the signaling game with the low type must almost always be broken up unsuccessfully for high δ. By $V_1^R(\delta) = g(\delta)V_1^A(\delta)$, the low type strictly prefers not to make an offer that is surely rejected. Also, he only makes a separating offer if this is accepted for sure.[16] As a

[15] This threshold is not necessarily identical with that of Claim B.1. For convenience we use, however, the same notation.

[16] More precisely, suppose the low type makes a separating offer with positive probability, which by Lemma B.1 yields V_1^{RS}. Then, we have $V_1^A(\delta) = V_1^{RS}$ and thus $V_1^R(\delta) < V_1^{RS}$, which implies from optimality that the offer must be accepted with probability one. (Otherwise, the agent would marginally lower the transfer.)

consequence, the low type must for high δ almost always offer some (semi-) pooling menu that is almost always rejected by the principal but that is also accepted with positive probability. From $\mu(\delta)\to 0$ there must then be for high δ a (semi-)pooling offer for which principals' posterior beliefs put almost probability one on the low type. As the low type must realize more than V_1^{RS} after acceptance to make this offer optimal, this contradicts Claim B.2.

To rule out the case where $\mu(\delta)\to 1$, note that for high δ the screening game with the high type would have to be broken up almost always. Given (A.3), it is intuitive, however, that this is not optimal for principals. (We omit a formal proof.) Finally, as at least one type is always successful in the screening game by Lemma B.2, the boundaries on $\mu(\delta)$ imply that also the aggregate break-up probabilities $\xi_i(\delta)$ must remain bounded away from one. □

Recall now the definition of g. In particular, recall that $V_i^P = V_i^R$ implies $V_i^R = gV_i^A$. If $\delta\to 1$ did not imply $g(\delta)\to 1$, we would have $m^A(\delta)\to 0$, $m^P(\delta)\to 1$, and thus $f(\delta)\to 1$. Hence, it would be almost costless for principals to wait for screening games, but it would still involve substantial costs for agents to wait for signaling games. Intuitively, this would imply $U^R > U^0$ for all high δ, for which the market would not be stationary.

Lemma B.4. *Along any sequence of equilibria where $\delta\to 1$ it holds that $g(\delta)\to 1$.*

Proof. Omitted.

We come now to a key result regarding the potential cross-subsidization of low types. If low types are cross-subsidized—say in the signaling game—and if their reservation value exceeds V_1^{RS} as $\delta\to 1$, the screening game will almost always be broken up with low types.

Lemma B.5. *Suppose that $V_1^R(\delta)\to \bar{V}_1^R > V_1^{RS}$ as $\delta\to 1$. Then it must hold that $\alpha_1^P(\emptyset,\delta)\to 1$.*

Proof. Define $\bar{C} = \{c\in C \mid U_2(c)\geqslant U^0 \text{and } V_2(c)\geqslant 0\}$, which by (A.1) and (A.4) is non-empty and compact. The following result is intuitive from (A.1)–(A.3).

Claim B.3. *All contracts $c\in B_2$ satisfy $c\in\bar{C}$. Moreover, there exists some finite $\bar{V}$ such that $V_i^R < \bar{V}$ holds for $i\in I$.*

Proof. Omitted.

We argue to a contradiction, implying that we can choose a sequence of equilibria (or an adequate subsequence) such that $\delta\to 1$, $\mu(\delta)\to\bar{\mu}$, $V_1^R(\delta)\to\bar{V}_1^R$ with $\bar{V}_1^R = V_1^{RS}+\bar{\varepsilon}_1$, and $\alpha_1^P(\emptyset,\delta)\to\bar{\alpha}_1^P(\emptyset)$ with $\bar{\alpha}_1^P(\emptyset) = 1-\bar{\varepsilon}_2$, where $\bar{\varepsilon}_1,\bar{\varepsilon}_2 > 0$. (Note that existence of a subsequence where $V_1^{\mathrm{R}}(\delta)$ converges follows from Claim B.3.) This implies existence of some $\bar{\delta}_1 < 1$ such that $V_1^{\mathrm{R}}(\delta) > V_1^{\mathrm{RS}} + \bar{\varepsilon}_1/2$ and $\alpha_1^{\mathrm{P}}(\emptyset, \delta) < 1 - \bar{\varepsilon}_2/2$ for $\delta > \bar{\delta}_1$.

Suppose now $\delta > \bar{\delta}_1$. By construction of the RS contracts and $V_1^{\mathrm{R}}(\delta) > V_1^{\mathrm{RS}} + \bar{\varepsilon}_1/2$, principals realize less than $U^0 - \bar{\varepsilon}_1/2$ with low types. Consequently, their offers to low types must be accepted with probability one. (Otherwise, i.e., in case of indifference, a marginal adjustment would ensure that the low type rejects, which would make a principal strictly better off.) Choose now the menu offer in the screening game that realizes the lowest payoff for the low type and denote the respective contracts by $c_i(\delta)$. The choice of $c_1(\delta)$, $\bar{\alpha}_1^{\mathrm{P}}(\emptyset) = 1 - \bar{\varepsilon}_2$ together with $V_1^{\mathrm{R}}(\delta) \to \bar{V}_1^{\mathrm{R}}$, and $g(\delta) \to 1$ due to Lemma B.4 ensure for any $\varepsilon > 0$ the existence of some $\bar{\delta}_3(\varepsilon) < 1$ such that for all $\delta > \bar{\delta}_3(\varepsilon)$ it holds that

$$V_1(c_1(\delta)) \leqslant V_1^{\mathrm{R}}(\delta) + \varepsilon. \tag{B.2}$$

(Otherwise, the low type could realize strictly more than $V_1^{\mathrm{R}}(\delta)$ by waiting for the screening game.) From (A.1) and (A.2) we can next two values $\bar{k} > 0$, $\underline{k} > 0$ such that

$$\left| \frac{dw_2(y)}{dy} \right| < \bar{k} \quad \text{and} \quad \frac{dv_2(2)}{dy} - \frac{dv_1(y)}{dy} > \underline{k} \tag{B.3}$$

for all $c \in \bar{C}$. By Claim B.3 this holds also for all $c_2(\delta)$. For some small ε we construct now a contract $c(\delta, \varepsilon)$ by transforming $c_2(\delta)$. For this choose $y(\delta, \varepsilon) = y_2(\delta) + \varepsilon \underline{k}$ and $t(\delta, \varepsilon)$ to ensure $V_2(c(\delta, \varepsilon)) = V_2(c_2(\delta))$. If all contracts $c(\delta, \varepsilon)$ are in $\bar{C}$—which can be confirmed for small ε—and if $\delta > \bar{\delta}_3(\varepsilon)$, we have by (B.2) and (B.3) that $V_1(c(\delta, \varepsilon)) < V_1^{\mathrm{R}}(\delta)$ and $U_2(c_2(\delta)) - U_2(c(\delta, \varepsilon)) < \bar{k} \varepsilon \underline{k}$. As a consequence, offering only $c(\delta, \varepsilon)$ realizes only marginally less with the high type but ensures rejection by the low type. As $U_1(c_1(\delta)) < U^0 - \bar{\varepsilon}_1/2$ this is strictly more profitable for all sufficiently low ε and all $\delta > \max\{\bar{\delta}_1, \bar{\delta}_2, \bar{\delta}_3(\varepsilon)\}$. □

Suppose now that $f(\delta) \to 1$ holds along a sequence where $\delta \to 1$. Then it follows from the definition of f in (6) and from Lemma B.3 that waiting for a screening game with a particular type becomes almost costless for principals. This allows to use arguments from the standard game of screening, where principals compete with public offers, to show the following. First, in the limit the low type cannot be cross-subsidized. As agents realize at least V_i^{RS} in the signaling game due to Lemma B.1 and as $g(\delta) \to 1$ by Lemma B.4, reservation values must then converge to the RS payoffs. By using the continuity of payoff functions it can then be finally shown that also allocations converge.

Lemma B.6. *Consider a sequence of equilibria where* $\delta \to 1$ *and* $f(\delta) \to 1$. *Then* $V_i^{\mathrm{R}}(\delta) \to V_i^{\mathrm{RS}}$ *and* $\pi(\delta) \to \pi^{\mathrm{RS}}$.

Proof. Omitted.

B.2. Proof of Proposition 2

Lemma B.7. *In any equilibrium with respective allocation π it holds that*

$$\mu^0 \int U_2(c)d\pi_2(c) + (1-\mu^0) \int U_1(c)d\pi_1(c) \geqslant U^0. \tag{B.4}$$

Proof. By principals' individual rationality, it must hold that

$$\begin{aligned}
&(1-b)\left[\mu \int_{c\in C} U_2(c)d\alpha_2^{\mathrm{A}}(c) + (1-\mu)\int_{c\in C} U_1(c)d\alpha_1^{\mathrm{A}}(c)\right] \\
&\quad + b\left[\mu \int_{c\in C} U_2(c)d\alpha_2^{\mathrm{P}}(c) + (1-\mu)\int_{c\in C} U_1(c)d\alpha_1^{\mathrm{P}}(c)\right] \\
&\quad + (1-b)[(1-\mu)\alpha_1^{\mathrm{A}}(\emptyset) + \mu\alpha_2^{\mathrm{A}}(\emptyset)]U^0 + b[(1-\mu)\alpha_1^{\mathrm{P}}(\emptyset) + \mu\alpha_2^{\mathrm{P}}(\emptyset)]U^0 \\
&\geqslant U^0.
\end{aligned}$$

Subtracting $U^0[\mu\xi_2 + (1-\mu)\xi_1]$ from both sides, while substituting μ from (3) and π_i from (2), this transforms to (B.4). □

Lemma B.8. *If $\{\pi_i\}_{i\in I} \in \bar{\Pi}$, then for any $i \in I$ and two contracts c' and c'' in B_i it must hold that $V_i(c') = V_i(c'')$, while for any $c_1 \in B_1$ and any $c_2 \in B_2$ it must hold that $V_1(c_1) \geqslant V_1(c_2)$ and $V_2(c_2) \geqslant V_2(c_1)$.*

Proof. Take some $\{\pi_i\}_{i\in I} \in \bar{\Pi}$ and a respective sequence of equilibria. Along a subsequence we have $V_i^{\mathrm{R}}(\delta) \to \bar{V}_i^{\mathrm{R}}$. (Finiteness follows from Claim B.1 in Lemma B.5.) We show first that $c \in B_1$ implies $V_1(c_1) \leqslant \bar{V}_1^{\mathrm{R}}$. (Recall that B_1 is the support of the limit distribution π_1.) We argue to a contradiction and suppose there exist $c \in B_1$ and $\bar{\varepsilon}_1 > 0$ such that $V_1(c) - \bar{V}_1^{\mathrm{R}} > \bar{\varepsilon}_1$. Then there exist $\bar{\varepsilon}_2 > 0$ and a neighborhood of c such that all contracts c' in this neighborhood satisfy $V_1(c') - \bar{V}_1^{\mathrm{R}} > \bar{\varepsilon}_2$, while for high δ either $\alpha_1^{\mathrm{A}}(\cdot,\delta)$ or $\alpha_1^{\mathrm{P}}(\cdot,\delta)$ put at least probability $\bar{\varepsilon}_2$ on this neighborhood. Using the definition of π_1 in (2), this follows by continuity of $V_i(\cdot)$, convergence of $\pi_1(\cdot,\delta)$, and as by Lemma B.3 $\xi_1(\delta)$ remains bounded away from one. As $g(\delta) \to 1$ by Lemma B.4, the low type's payoff from waiting to implement these contracts would exceed $V_1^{\mathrm{R}}(\delta)$ for high δ—a contradiction. The argument for $V_1(c_1) \leqslant \bar{V}_2^{\mathrm{R}}$, $i = 2$, and incentive compatibility is analogous. □

Note next that by Lemmas B.1 and B.4 any contract $c \in B_i$ for $\{\pi_i\}_{i\in I} \in \bar{\Pi}$ must satisfy $V_i(c) \geqslant V_i^{\mathrm{RS}}$. Consider finally the following strategies for the (one-shot) signaling game. The agent offers the menu $B_1 \cup B_2$, the principal accepts, and type i randomizes over contracts according to the probabilities specified by π_i. Furthermore, the principal has pessimistic beliefs for all other menus. As (B.4) must also hold in the limit, optimality follows from Lemmas B.7, B.8 and $V_i(c) \geqslant V_i^{\mathrm{RS}}$.

B.3. Proof of Proposition 3

Lemma B.9. *There exists $\bar{b} < 1$ such that for all $b > \bar{b}$ it must hold for any sequence of equilibria supporting some $\pi \in \bar{\Pi}$ that $f(\delta) \to 1$.*

Proof. We first show that for all $b > \bar{b}$ it holds that $V_1^{\mathrm{R}}(\delta) \to V_1^{\mathrm{RS}}$ as $\delta \to 1$. Take a sequence where $V_1^{\mathrm{R}}(\delta)$ remains bounded away from V_1^{RS} as $\delta \to 1$. By Lemmas B.1 and B.4, V_1^{RS} represents a lower boundary for the limit, implying that we only have to consider the case where $V_1^{\mathrm{R}}(\delta)$ exceeds V_1^{RS} also in the limit. By Lemma B.5 this implies $\alpha_1^{\mathrm{P}}(\emptyset, \delta) \to 1$ such that by Lemma B.2 $\alpha_2^{\mathrm{P}}(\emptyset, \delta) = 0$ holds for all high δ. Moreover, by Lemma B.3 $\mu(\delta) > \Delta/2$ holds for all high δ. Using the definition of $\mu(\delta)$ in (3), $\alpha_2^{\mathrm{P}}(\emptyset, \delta) = 0$, and $\alpha_1^{\mathrm{P}}(\emptyset, \delta) \to 1$, $\mu(\delta) > \Delta/2$ requires

$$b < \bar{b} := \frac{\mu^0(1 - \Delta/2)}{\mu^0(1 - \Delta/2) + (\Delta/2)(1 - \mu^0)}.$$

Thus, by way of contradiction we have shown $V_1^{\mathrm{R}}(\delta) \to V_1^{\mathrm{RS}}$ for $b > \bar{b}$. Arguing once more to a contradiction, suppose next for $b > \bar{b}$ that $\delta \to 1$ and $V_1^{\mathrm{R}}(\delta) \to V_1^{\mathrm{RS}}$, while also $f(\delta) \to \bar{f} < 1$. By definition of U^{R} and as surely $U^{\mathrm{P}}(\delta) \geqslant U^{\mathrm{A}}(\delta)$, $U^{\mathrm{R}} = U^0$ implies existence of $\bar{\varepsilon}_1 > 0$ such that $U^{\mathrm{P}}(\delta) > U^0 + \bar{\varepsilon}_1$ for all high δ. Recall next that by Lemmas B.1 and B.4, $V_i^{\mathrm{R}}(\delta)$ is bounded from below by V_i^{RS} as $\delta \to 1$. By construction of the RS contracts, $U^{\mathrm{P}}(\delta) > U^0 + \bar{\varepsilon}_1$ requires then that (i) c_2^{RS} is not first-best and that (ii) the principals' offer is made more efficient by providing a sufficiently large cross-subsidization for the low type. Formally, there exists $\bar{\varepsilon}_2 > 0$ such that $V_1^{\mathrm{P}}(\delta) > V_1^{\mathrm{RS}} + \bar{\varepsilon}_2$ for all high δ. As principals always offer such menus due to $U^{\mathrm{P}}(\delta) > U^{\mathrm{R}}$ and as $g(\delta) \to 1$ holds by Lemma B.4, $V_1^{\mathrm{R}}(\delta)$ would remain bounded away from V_1^{RS}, which yields a contradiction. □

Assertion (i) of Proposition 3 follows now from Lemmas B.9 and B.6. Turn next to assertion (ii), which follows immediately from the definition of interim efficiency and the results obtained in the proof of Proposition 2. To see this, recall that $\pi \in \bar{\Pi}$ implies $V_i(c) = V_i(c')$ for all c and c' in B_i and $V_i(c) \geqslant V_i(c')$ for all $c \in B_i$ and $c' \in B_j$. Moreover, (B.4) holds also in the limit, i.e., for $\pi \in \bar{\Pi}$. Finally, by Lemmas B.1 and B.4 any contract c in the support of π_i must satisfy $V_i(c) \geqslant V_i^{\mathrm{RS}}$.

B.4. Proof of Proposition 1

For given values $V_i^{\mathrm{R}} \geqslant 0$ and μ, we define the program $\Lambda^+(V_1^{\mathrm{R}}, V_2^{\mathrm{R}}, \mu)$ as follows: Contracts $c_i \in C$ are chosen to maximize $\mu U_2(c_2) + (1 - \mu) U_1(c_1)$ subject to $V_1(c_1) \geqslant V_1(c_2)$ and $V_i(c_i) \geqslant V_i^{\mathrm{R}}$ for $i \in I$. By (A.1) and (A.2), $\Lambda^+(\cdot)$ has a solution, while optimality implies $V_2(c_2) = V_2^{\mathrm{R}}$. We denote the realized payoff by $U^+(\cdot)$. The program can have multiple solutions, obtaining different utilities for $i = 1$. Denote by $V_1^+(\cdot)$ the (compact and convex) set of lotteries over the realized utilities for $i = 1$.

Lemma B.10. *$U^+(\cdot)$ is continuous, while $V_1^+(\cdot)$ is upper-semicontinuous.*

Proof. We denote the set of solutions $(c_1, c_2) \in C^2$ for $\Lambda^+(\cdot)$ by $C^+(\cdot)$. Optimality implies $x_1 = x_1^{\mathrm{RS}}$, $t_2 = V_2^{\mathrm{R}} - v_2(x_2)$, and that one of the constraints for $i = 1$ binds such that $t_1 = \max\{V_1^{\mathrm{R}}, V_1(c_2)\} - v_1(x_1)$. After substitution, the objective function is continuous in x_2 and in the parameters $(V_1^{\mathrm{R}}, V_2^{\mathrm{R}}, \mu)$, implying continuity of $U^+(\cdot)$ and upper-semicontinuity of $C^+(\cdot)$. Upper-semincontinuity of $V_1^+(\cdot)$ follows again from the continuity of payoff functions.[17] □

Define next the program $\Lambda^-(V_1^{\mathrm{R}}, V_2^{\mathrm{R}}, \mu)$ as follows: $c \in C$ is chosen to maximize $\mu U_2(c) + (1-\mu)U^0$ subject to $V_1(c) \leqslant V_1^{\mathrm{R}}$ and $V_2(c) \geqslant V_2^{\mathrm{R}}$. Denote the realized utility by $U^-(\cdot)$. The proof of the following result is analogous to that of Lemma B.10.

Lemma B.11. *$U^-(\cdot)$ is continuous.*

We now specify a candidate equilibrium. In the signaling game, the high type offers c_2^{RS}, while the low type offers a non-acceptable contract with probability $\alpha_1^{\mathrm{A}}(\emptyset)$ and c_1^{RS} with the residual probability. (For the moment we omit the indexation with δ.) In the screening game, principals choose a solution to Λ^- with probability $\alpha_1^{\mathrm{P}}(\emptyset)$ and solutions to Λ^+ with the residual probability. We consider now a fixed point problem in $\alpha_1^{\mathrm{A}}(\emptyset)$, $\alpha_1^{\mathrm{P}}(\emptyset)$, $m^{\mathrm{P}} \in [0, 1]$, and $V_1^{\mathrm{R}} \in [0, \bar{V}]$. (Note that, by the properties of the matching function, we can for any m^{P} determine m^{A} according to some function $\tau(m^{\mathrm{P}})$ that is continuous and strictly decreasing with $\tau(0) = 1$ and $\tau(1) = 0$.) For given choices of these variables, and given δ, we determine f and g from (6), $V_2^{\mathrm{R}} = gV_2^{\mathrm{RS}}$, and as in (2)

$$\mu = \frac{\mu^0[(1-b)(1-\alpha_1^{\mathrm{A}}(\emptyset)) + b(1-\alpha_1^{\mathrm{P}}(\emptyset))]}{1-\mu^0[1-(1-b)(1-\alpha_1^{\mathrm{A}}(\emptyset)) - b(1-\alpha_1^{\mathrm{P}}(\emptyset))]}. \tag{B.5}$$

Define now a mapping $\varphi(\cdot)$ as follows. A tuple $(\hat{\alpha}^{\mathrm{A}}(\emptyset), \hat{\alpha}^{\mathrm{P}}(\emptyset), \hat{m}^{\mathrm{P}}, \hat{V}_1^{\mathrm{R}})$ is an element of $\varphi(\alpha^{\mathrm{A}}(\emptyset), \alpha^{\mathrm{P}}(\emptyset), m^{\mathrm{P}}, V_1^{\mathrm{R}})$ if the following conditions hold:

Condition 1: If $\frac{\delta b}{1-\delta[1-b]}\max\{U^+(\cdot), U^-(\cdot)\} \leqslant U^0$, set $\hat{m}^{\mathrm{P}} = 1$. Otherwise, $\hat{m}^{\mathrm{P}}$ solves

$$\frac{\delta \hat{m}^{\mathrm{P}} b}{1-\delta[1-b\hat{m}^{\mathrm{P}}]}\max\{U^+(\cdot), U^-(\cdot)\} = U^0. \tag{B.6}$$

[17] Observe that $C^+(\cdot)$ may not be convex. For this reason, we allow firms to randomize so as to realize any utility in the convex set $V_1^+(\cdot)$. Alternatively, it can be checked that (A.1) and (A.2) together with $d^2v_2(x)/dx^2 \leqslant d^2v_1(x)/dx^2$ convexifies the set of solutions.

Condition 2:

$$\hat{\alpha}_1^{\mathrm{P}}(\emptyset) = \begin{cases} 1 & \text{if } U^-(\cdot) > U^+(\cdot), \\ 0 & \text{if } U^-(\cdot) < U^+(\cdot), \\ \in [0,1] & \text{if } U^-(\cdot) = U^+(\cdot). \end{cases} \tag{B.7}$$

Condition 3:

$$\hat{\alpha}_1^{\mathrm{A}}(\emptyset) = \begin{cases} 1 & \text{if } V_1^{\mathrm{RS}} < V_1^{\mathrm{R}}, \\ 0 & V_1^{\mathrm{RS}} > V_1^{\mathrm{R}}, \\ \in [0,1] & V_1^{\mathrm{RS}} = V_1^{\mathrm{R}}. \end{cases} \tag{B.8}$$

Condition 4: $\hat{V}_1^{\mathrm{R}}$ is determined by the requirement that there exists $V_1 \in V_1^+(\cdot)$ such that[18]

$$\hat{V}_1^{\mathrm{R}} = \max\left\{ \frac{\delta m^{\mathrm{A}}[b(1-\alpha_1^{\mathrm{P}}(\emptyset))V_1 + (1-b)V_1^{\mathrm{RS}}]}{1-\delta[1-m^{\mathrm{A}}(b(1-\alpha_1^{\mathrm{P}}(\emptyset))+1-b)]}, \frac{\delta m^{\mathrm{A}} b(1-\alpha_1^{\mathrm{P}}(\emptyset))V_1}{1-\delta[1-m^{\mathrm{A}}b(1-\alpha_1^{\mathrm{P}}(\emptyset))]} \right\}. \tag{B.9}$$

As $\varphi(\cdot)$ is convex and compact and by Lemmas B.10 and B.11 upper-semicontinuous, while the compact domain is not smaller than the range, we can apply Kakutani's fixed point theorem. For given δ, we pick a fixed point and index the respective variables by δ. We derive next some results for this sequence of candidate equilibria.

Lemma B.12. *Along the constructed sequence it holds that* $f(\delta) \to 1$.

Proof. We argue to a contradiction. We can then choose a subsequence where $f(\delta) \to \bar{f} < 1$, while from $\delta \to 1$ we have $g(\delta) \to 1$, $V_2^{\mathrm{R}}(\delta) \to V_2^{\mathrm{RS}}$, and $V_1^{\mathrm{R}}(\delta) \to \bar{V}_1^{\mathrm{R}} \geqslant V_1^{\mathrm{RS}}$. Also, for high δ we have that $m^{\mathrm{P}}(\delta) < 1$ is determined by (B.6).

Claim B.1. *If* $f(\delta) \to \bar{f} < 1$, *then* $\bar{V}_1^{\mathrm{R}} > V_1^{\mathrm{RS}}$.

Proof. We argue to a contradiction and assume $\bar{V}_1^{\mathrm{R}} = V_1^{\mathrm{RS}}$. By $V_2^{\mathrm{R}}(\delta) \to V_2^{\mathrm{RS}}$ and construction of the RS contracts it is immediate that $U^-(\delta)$ cannot remain bounded away from U^0 as $\delta \to 1$. To satisfy (B.6) this implies $U^+(\delta) > U^-(\delta)$ for high δ and thus, using (B.7), that $\alpha_1^{\mathrm{P}}(\emptyset, \delta) = 0$. Moreover, as $U^+(\delta)$ must exceed U^0 by a sufficiently large margin so that $f(\delta)U^+(\delta) = U^0$ in (B.6) holds with $\bar{f} < 1$, it follows again from construction of the RS contracts that $(c_1, c_2) \in C^+(\cdot)$ must involve some non-marginal cross-subsidization. But if $V_1(\delta)$ remains bounded way from V_1^{RS} this together with $g(\delta) \to 1$ and $\alpha_1^{\mathrm{P}}(\emptyset, \delta) = 0$ implies from (B.9) that $V_1^{\mathrm{R}}(\delta)$ remains bounded away from V_1^{RS}—a contradiction. □

[18] Note that if $\hat{\alpha}_1^{\mathrm{P}}(\emptyset) = 1$, then the choice of $V_1 \in V_1^+(\cdot)$ does not influence (B.9) at all.

With $V_1^R(\delta) \to \bar{V}_1^R > V_1^{RS}$ it is next intuitive that $U^-(\delta) > U^+(\delta)$ holds for all high δ. But by (B.9) this contradicts $V_1^R(\delta) > V_1^{RS}$.[19] □

Lemma B.13. *Along the constructed sequence it holds that $V_i^R(\delta) \to V_i^{RS}$ for $i \in I$.*

Proof. Given $f(\delta) \to 1$ from Lemma B.12, it is straightforward to show that $g(\delta) \to 1$. Otherwise, given $V_2^R(\delta) = g(\delta) V_2^{RS}$ and the definition of $V_1^R(\delta)$ in (B.9), it is straightforward that (B.6) could not hold. (That is, we would have $U^R > U^0$.)[20] From $g(\delta) \to 1$ we have $V_2^R(\delta) \to V_2^{RS}$, while from (B.9) V_1^{RS} is in the limit a lower boundary for $V_1^R(\delta)$. If $V_1^R(\delta)$ remained bounded away (from above), it is again straightforward that $U^+(\delta) > U^-(\delta)$ holds for high δ and thus that $\alpha_1^P(\emptyset, \delta) = 0$, which by (B.9) yields a contradiction. □

Note now that, by Lemma B.13 and construction of the RS payoffs, we can indeed neglect for high δ the high type's incentive compatibility constraint in Λ^+. But then it follows by construction of the fixed point that we indeed constructed an equilibrium.[21] Finally, by Lemma B.12 convergence to the RS allocation follows then from Lemma 6.

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[19] Formally, it holds by $g(\delta) \to 1$, $V_1^R(\delta) \to \bar{V}_1^R > V_1^{RS}$, and (B.9) in Condition 4 that $V_1^R(\delta) \to V_1(\delta) \in V^+(\cdot)$. Given this choice of $V_1^R(\delta)$ (in the limit), $U^-(\delta) > U^+(\delta)$ is then immediate from $\bar{V}_1^R > V_1^{RS}$.

[20] This is similar to Lemma B.4, though now the assertion does *not* refer to a sequence of equilibria but to the sequence of constructed *candidate* equilibria.

[21] Note that principals have no better alternative in the screening game. In the signaling game, we can specify pessimistic out-of-equilibrium beliefs to support the strategies.

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