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# Marx, the Production Function and the Old Neoclassical Equilibrium: Workable under the Same Assumptions? With an Appendix on the Likelihood of Reswitching and of Wicksell Effects*

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**Abstract**
A stochastic approach has been introduced to explain the empirically observed fact that wage curves calculated from input-output systems tend to be nearly linear and that the paradoxes of capital appear to be rare. The stochastic approach allows to justify the simplifying treatment of normal prices common to 19th and early 20th century authors as diverse as Marx (transformation problem), Wicksell (old neoclassical equilibrium), J.B. Clark (neoclassical production function). It is shown that the likelihood of reverse capital deepening is much lower than that of Wicksell effects. With this, the likely characteristics of the wage frontier obtained from a multiplicity of input-output tables are derived. The conclusion summarises what we know and do not know about the validity of the Cambridge critique of capital.

**Keywords**: Capital theory, Random matrices, Aggregate Production Function, Transformation problem, Wicksell effects

**JEL Codes**: B13, B14, E25

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## 1. Long-period positions: a stochastic approach

The critique of capital advanced by the Cambridge economists affected both neoclassical and Marxian theories. The aim of this paper is to analyse what remains of the critique if the physical data of the theory are regarded as stochastic magnitudes. The question has already been pursued for the aggregate production function in Schefold (2013a) and for Marx in Schefold (2014). Here, the critique is extended to what we call the old neoclassical equilibrium and then again applied to the production function in a new form. The old conception must first be differentiated from other neoclassical models.

Most economists of the 19th century and beyond shared the conviction that not only prices of homogeneous commodities and of homogeneous non-produced factors of production tend to get uniform, if the process of competition is not disrupted, but also the rates of profit on the cost of capital advanced. They analysed economic development, which is an evolutionary process in which the data like natural resources, technical knowledge and mental attitudes change slowly, by taking these data for so-called long-period positions as given and fixed. The art of the approach consisted in a periodization such that the time for convergence towards uniform prices was sufficiently long to lend credibility to the assumption that the tendency had become reality; change then was analysed on the assumption that the forces engendering change could be understood by comparing different long-period positions. Various authors, in particular Petri (2004), following in the footsteps of Garegnani (1960) and of Sraffa (1960), have shown how this method was used both by the classical economists, who started from a determination of distribution by means of a given real wage, *and* by neoclassical economists who determined distribution by means of supply and demand for labour and capital, where capital had to be conceived as given in terms of an amount of value, and the amounts of capital goods used in the several lines of production were determined endogenously; their total value corresponded to the value of the capital endowment.

Clark described this conception succinctly and insisted on its Ricardian origins: „We have now before us a picture of a static industrial world … it produces and consumes wealth; but the kinds of wealth that it creates and uses, and the quantity that it creates of all the various kinds remain unchanged. … values are here ‚natural' in the Ricardian sense, for everything sells at its ‚cost of production' and no *entrepreneur* makes a profit." (Clark 1896, pp. 399-400) Profit here means surplus profit, for capital earns interest at a normal rate. We can see how Clark uses the classical theory of prices, with uniform rates of wages, of profits (interest) and rents for equal kinds of land in a neoclassical context, i.e. explaining distribution by means of marginal productivity theory.

This 'old neoclassical equilibrium' approach contrasts with that of modern intertemporal general equilibrium theory, where the endowments are given as quantities of capital goods and non-produced means of production, with the consequence that the rate of profit can become uniform only as a tendency over many periods (so-called turnpike theorem by Dorfman, Samuelson and Solow 1958, see Schefold 1997, pp. 425–501). Among the older neoclassical economists only Walras had a vector of endowments of capital goods inherited from the past and of non-produced factors at the beginning of a long-period position, which he associated with normal prices. That this was a mistake has been pointed out by Gareganani (1960). It has been analysed in more detail by Eatwell (1987) and Petri (2004), among others. The Walrasian model of capital formation was formalised by Morishima with inequalities. Morishima showed that an equilibrium existed under rather general conditions, but this could also involve degenerate solutions, involving no reproduction of capital at all (Schefold 2015). The Walras-Morishima model therefore is not successful as a representation of how economies can reach steady reproduction, starting from arbitrary initial conditions, but it remains interesting as a reference case.

Böhm-Bawerk had an intertemporal model with a uniform rate of profit, because, unlike modern intertemporal theory, he did not assume a given vector of endowments. Instead, he assumed that the supply of capital was given in the form of a subsistence fund. The fund included the subsistence fund both for the workers engaged in current production and for those engaged in the manufacture of the means of production. Direct and indirect labour thus was employed so that a uniform rate of profit could result. The solution of the old neoclassical economists of taking the quantity of capital as given in the form of a value magnitude was simpler and has survived until today in the form of the production function which is introduced as a one-sector model, but then applied to the economy as a whole.

The Cambridge debate later attempted to show that the aggregation of capital, with the aim of reproducing the results of the one-sector model, was in general impossible because of paradoxes in the valuation of capital, in particular reverse capital deepening. The critique of the old neoclassical equilibrium model and that of the production function must be similar because both use the concept of aggregate capital, as will be confirmed in this paper.

Meanwhile, it has been found that the paradoxes of capital seem to occur only rarely in empirical investigations (Han and Schefold 2006), while many neoclassicals and Marxist authors disregard this critique often without knowing it. This reluctance to face objections is, to say the least, not always reasonable, but it also has rational causes. There are exceptions to the critique. It is generally accepted that it does not apply to a one-commodity world. Why should the economy as a whole not behave by and large as a one-commodity world? A more accurate investigation seems to be needed to determine the conditions under which the paradoxes can appear or not appear. It turns out that a stochastic approach changes the picture, as I have shown in three publications: Linear wage curves as in one-commodity models result, if prices are expressed in terms of the standard commodity which is Han eigenvector of the input matrix or if the labour theory of value holds, with equal organic compositions of capital, which is the case if the labour vector happens to be Han eigenvector. It had not been shown before that simple wage curves could also result from special forms of the input matrix itself. In brief: If the matrices have random properties and if the numéraire vector and the labour vector also stand in certain random relationships to each other, the wage curves tend to be straight lines. In Schefold (2013a), sufficient conditions are given for the construction of approximate surrogate production functions, extending the realm in which this is possible from one-commodity world to more realistic conditions. On the other hand, Schefold ( 2013b) shows that only a few of a multiplicity of linear wage curves will appear on the envelope, if the position of the wage curve is random; hence the possibilities for substitution are not as ample as the neoclassical theorists assume. Schefold (2014) shows that a central proposition of Marx in his transformation of values into prices of production holds under analogous conditions: profits equal total surplus value.

I shall here summarise these results, adding a clarification of the method, and showing that a similar argument can be made to provide a partial justification of the old neoclassical general equilibrium model, which used the idea of capital as an endowment of a quantity of value. It will be seen that the conditions to establish the Marxian result are somewhat less restrictive than those required for the old neoclassical general equilibrium model and for the production function. It turns out that, by and large, random properties of the systems of production suffice to justify the conceptions of capital of the 19$^{th}$ century economists, i.e. of those using the old neoclassical equilibrium, the production function and Marxian theory. By contrast, the implications for the critique of general intertemporal equilibrium theory remain to be investigated in the future.

This is not primarily a contribution to applied economics, as some thought, when they first saw one or another of the three papers mentioned above. It is primarily an exercise in pure theory. Just as Sraffa shows that the wage curve will be linear under a stated assumption (if the numéraire is equal to the standard commodity), it is here shown that the wage curve tends to be linear with probability one, if certain stochastic conditions are met. To show this with full mathematical rigour would require a very complicated paper. Like Sraffa I shall not use the most advanced mathematical methods to present the argument. As in Sraffa and as in classical and old neoclassical economics, long-period positions are assumed, without providing explicit models for the gravitation of market prices to normal prices. As John Bates Clark puts it: “In the midst of all changes there are at work forces that fix rates to which, at any one moment, wages and interest tend conform … What would be the rate of wages, if labour and capital were to remain fixed in quantity, if improvements in the mode of production were to stop, if the consolidating of capital were to cease and if the wants of consumers were never to alter?" (Clark 1899, p. vi). Assumptions in economic theory are always to some extent counterfactual. We not only work with models in which we assume that the uniformity of the rates of remuneration has been obtained, but also with single product systems (although I spent so many years on the analysis of joint production). We need the randomness assumption concerning the input matrix and various covariance-assumptions concerning the numéraire vector, the labour vector and the vector for the composition of output or the surplus. Like the assumptions about the uniformity of prices, the assumptions about the stochastic properties cannot be expected to be fulfilled perfectly in actual reality. As we can observe different rates of profits in different sectors and believe in the convergence towards a uniform rate of profit, we can observe that indicators of the stochastic properties of the system are not proof of a perfect fulfilment of the conditions, but of a tendency towards such fulfilment. Research on this has only begun. Indicators of such fulfilment are, among others, the variance of the input-output coefficients in actual input-output tables, considered at various levels of aggregation and the spectrum of the eigenvalues.[1]

## 2. Economic intuition and the stochastic approach

Readers less interested in the mathematical background may omit this section. The problem of empirical application can only be touched upon in this paper, but I briefly try to reply to some objections, which have been made to the earlier papers in letters and discussions. Some of these objections concerned the use of input-output tables as a proxy to represent the spectrum of techniques used in the theory. They are in fact more aggregated than the theory ideally would require. On the other hand, the theory is not concerned with the individual commodities, as produced by individual artisans according to the specifications of individual consumers. But then one would be concerned with the market prices contracted in individual transactions; to begin at a certain intermediate level of aggregation is inherent in the conception of natural prices. It has also been objected that input-output tables do not describe individual methods of production; they concern average techniques. In this, they are closer to Marx with his conception of socially necessary labour time than to the dominant, cost minimising techniques considered in classical theory and also in most interpretations of Sraffa

[1] The spectrum of the eigenvalues has been examined for the input-output tables of a number of economies by Mariolis and Tsoulfidis, see in particular Mariolis and Tsoulfidis (2014). I owe special gratitude to Anwar Shaikh for a provisional analysis of disaggregation on the variance of the input-output coefficients and for discussions concerning the theory of prices in random systems, see his forthcoming book (Shaikh 2015). I also have to thank for special advice on the mathematical properties of random matrices by Professors Joachim Weidmann and Götz Kersting, both in the Mathematical Faculty of my University.

(Schefold 1988). Here we are concerned with their stochastic properties. Schefold (2013a) bases the argument on a theorem by Goldstein and Neumann which states that the subdominant eigenvalues of a semi-positive matrix, the input coefficients of which are independently and identically distributed with a mean characteristic for each industry, will, if certain conditions on the variance of the coefficients in each row and on the covariance in comparison between rows (the rows stand for the industries) are met, tend to zero with probability one, as the number of the sectors tends to infinity, with the dominant eigenvalue being kept constant. Loosely speaking: All eigenvalues except the dominant eigenvalue will be small in modulus for large matrices, if the coefficients in each industry are random, with a mean specific for the industry.

Input-output matrices are in fact fairly large; hence the effect of randomness on the spectrum of eigenvalues should be visible, and it is. In order to see how, assume that the matrix is diagonalisable. We then get, for a matrix of order $n$, $n$ eigenvalues $\mu_1,...,\mu_n$, which we can assume to be ordered according to modulus, such that $\mu_1 > |\mu_2| \geq ... \geq |\mu_n| \geq 0$, where we also assume that the matrix is imprimitive and $\mu_1 = dom\,\mathbf{A}$, $A \geq 0$ and $\mathbf{A}$ indecomposable. The difference $\mu_1 - |\mu_2|$ is often called the spectral gap. We measure it in percentage terms: $(\mu_1 - |\mu_2|)/\mu_1$. The theorem by Goldstein and Neumann therefore says that the spectral gap for what they define as random matrices will tend to 100 % as $n \to \infty$. The empirical analyses by Mariolis and Tsoulfidis and by Anwar Shaikh do not indicate that the spectral gap rapidly tends to 100 % for empirical input-output matrices.[2] Rather, one finds that the gap is around 50 %, but it seems to increase with disaggregation according to an example by Anwar Shaikh and, what is more important, the remaining eigenvalues tend to zero quite rapidly after a handful of the first few which tend to zero more slowly. The implication of this finding has been discussed in Schefold (2013a): This small number of eigenvalues with significant modulus can give rise to wiggles of wage curves which otherwise turn out to be stretched hyperbolas approximating linearity.

The following is a mathematical result for the spectral gap which I have not yet seen quoted in the economic literature (Haveliwala and Kamvar 2003): *If* $\mathbf{A} = \gamma\mathbf{P} + (1-\gamma)\mathbf{ce}$, *where* $\mathbf{P}$ *is a stochastic matrix* $\mathbf{P} \geq 0, \mathbf{eP} = 0$, *where* $\mathbf{c}$ *is a positive column vector and* $\mathbf{e} = (1,...,1)$ *a row vector with* $\mathbf{ec} = 1$, *the modulus of the second eigenvalue is* $|\mu_2| \leq \gamma$. $\mathbf{P}$ can be interpreted as a probability matrix and $\mathbf{c}$ as a probability distribution. We therefore get a result which is related to that of Goldstein and Neumann. The coefficients of $\mathbf{ce}$ on each row are all equal and therefore equal to their mean; instead of having this mean given and the coefficients being different with a certain variance, as in

[2] Mariolis and Tsoulfidis and other authors work with vertically integrated industries, hence with matrix $\mathbf{H} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{A}$. If $\lambda$ is an eigenvector of $\mathbf{A}$, $(1-\lambda)^{-1}\lambda$ is the corresponding eigenvalue of $\mathbf{H}$. Hence the spectral gap is large for productive matrices $\mathbf{A}$ with $|\lambda|<1$, if and only if it is large for $\mathbf{A}$.

Goldstein and Neumann, they are equal but the coefficients of the input matrix $\mathbf{A}$, which interests us, are each augmented by a matrix of perturbations. These perturbations are all positive and represent a probability distribution.

This confirms an assertion made in Schefold (2013a): our $\mathbf{A}$ matrices can be interpreted as determinate matrices of a very simple structure, namely $\mathbf{A}$ equals $\mathbf{ce}$ where $\mathbf{c} > 0$ and $\mathbf{ec} = \mu_1$ is the dominant root of $\mathbf{A}$, but this simple structure is disturbed by a matrix of perturbations which, in the case of Haveliwala and Kamvar, is semi-positive. This latter property represents a drawback, for we are interested in input-output structures where some, perhaps many, inputs are zero. The variance condition by Goldstein and Neumann allows this to happen, but not the formulation by Haveliwala and Kamvar. It would be desirable to modify their theorem accordingly. Moreover, the considerations in Schefold (2013a) suggest that a generalisation must be possible. The matrix $\mathbf{ce}$ which is being disturbed can be replaced by a matrix $\mathbf{cf}$, with any $\mathbf{f} > 0$, representing a different distribution from that indicated by $\mathbf{e}$, and $\mathbf{f}$ was interpreted as a leading industry in Schefold (2013a). These are therefore future extensions. In this paper, we stick to random matrices with coefficients which are distributed independently and identically on each row. One reason concerns the choice of technique. If a random system is given, with $\mathbf{A}$ approximately equal to $\mathbf{ce}$ and yielding an approximately linear wage curve, the replacement of one method by another, which is more profitable at a given rate of profit, will again lead to an approximately linear wage curve, if the input coefficients of the other method have the same distribution. This is most plausible, if all rows are i.i.d.[3]

I often hear the objection that input coefficients are not random. “Cars have four wheels!”, it is said. But cars can have six wheels, if they are lorries, and, whether such interdependences show, is, up to a point, a question of aggregation. The tallness of people is regarded as a random variable in a population, although there are twins.

It is also objected that random matrices of the type used by Goldstein and Neumann (henceforth to be called random matrices of type $\mathbf{ce}$) would have to show a very even distribution of input coefficients along each row (for each industry), and this seems not to be realistic. But the variance in the theorem is large enough to allow zeros in the matrix. It cannot be a sparse matrix, but it can have many zeros, if other coefficients are correspondingly larger. This is a loose formulation, but perhaps apt to quell doubts based on a misleading intuition. Random matrices of type $\mathbf{ce}$, of given order $n$, can

[3] Salvadori and Steedman (1988) argued that not only reswitching but even switching was not possible among techniques with uniform compositions of capital and hence linear wage curves, because the combined techniques would not be of equal capital composition. This argument does not apply here, because the causes for the linearity of the wage curve are different. Here, we neither assume that prices are standard prices, nor that they are equal to values, but it is the structure of $\mathbf{A}$ that counts.

exhibit a considerable variance of the coefficients on each row, although all non-dominant eigenvalues are small.[4]

[4] In fact, we shall see in section 2 that the main property of the input matrix needed to obtain nearly straight wage curves is a large spectral gap; that the matrix be random is only the most interesting sufficient condition to get this result. Perhaps the following intuitive argument will convince the reader that the elements of the input matrix may vary considerably, even if the non-dominant eigenvalues are small. Consider an input matrix $\mathbf{A}$ of order $n$ fulfilling the same assumptions as above (semi-positive, indecomposable and diagonalisable). Hence there is an invertible matrix $\mathbf{T}, \mathbf{T}^{-1} = \mathbf{G}$, such that

$$\mathbf{TAT}^{-1} = \mathbf{D} = diag\{\mu_1, \mu_2, ..., \mu_n\} ,$$

where $\mathbf{D}$ is a diagonal matrix with the eigenvalues on the diagonal and $\mu_1 = dom\,\mathbf{A}$. Hence also $\mathbf{AT}^{-1} = \mathbf{T}^{-1}\mathbf{D}$ and $\mathbf{T}^{-1}\mathbf{DT} = \mathbf{A}$. The rows of $\mathbf{T}$, $\mathbf{t}_i$, and the columns of $\mathbf{G}$, $\mathbf{g}^j$, are orthogonal and $\mathbf{t}_i\mathbf{g}^j = \delta_{ij}$, that is: the scalar product of the rows of $\mathbf{T}$ and the columns of $\mathbf{G}$ are zero, if row and column belong to different eigenvalues, and equal to one, if they belong to the same eigenvalue. Thus we can write

$$\begin{aligned} \mu_1\mathbf{g}^1\mathbf{t}_1 + ... + \mu_n\mathbf{g}^n\mathbf{t}_n &= \mathbf{GT}[\mu_1\mathbf{g}^1\mathbf{t}_1 + ... + \mu_n\mathbf{g}^n\mathbf{t}_n] \\ &= \mathbf{G}\begin{bmatrix} \mu_1\mathbf{t}_1 \\ \vdots \\ \mu_n\mathbf{t}_n \end{bmatrix} = \mathbf{GDT} = \mathbf{A} \\ &= \mu_1\mathbf{M}_1 + ... + \mu_n\mathbf{M}_n. \end{aligned}$$

$\mathbf{M}_i = \mathbf{g}^i\mathbf{t}_i$ are idempotent matrices of rank $1$ and orthogonal in the following sense:

$$\mathbf{M}_i\mathbf{M}_i = (\mathbf{g}^i\mathbf{t}_i)(\mathbf{g}^i\mathbf{t}_i) = \mathbf{g}^i(\mathbf{t}_i\mathbf{g}^i)\mathbf{t}_i = \mathbf{g}^i\mathbf{t}_i = \mathbf{M}_i; \mathbf{M}_i\mathbf{M}_j = 0\ (i \neq j) .$$

Matrix $\mathbf{A}$ therefore is a linear combination of $n$ matrices, each of rank 1, with the eigenvalues as coefficients: $\mathbf{A} = \mu_1\mathbf{M}_1 + ... + \mu_n\mathbf{M}_n$. The converse is also true, as one can show easily: If a system of matrices $\mathbf{M}_i$ of rank 1 is given, $\mathbf{M}_i$ of the form $\mathbf{g}^i\mathbf{t}_i$, with $\mathbf{g}^i$ and $\mathbf{t}_i$ being a column and a row vector in $n$ dimensional complex space, respectively, and if the matrices fulfil the orthogonality conditions, the vectors and $\mathbf{t}_i$ and $\mathbf{g}^i$ turn out to be the eigenvectors pertaining to given complex eigenvalues $\mu_1, ..., \mu_n$ of a matrix $\mathbf{A}$ defined by $\mathbf{A} = \mu_1\mathbf{M}_1 + ... + \mu_n\mathbf{M}_n$.

We now can see how examples of matrices with small non-dominant eigenvalues and yet with considerable variance can be generated. The difficulty is to make sure that they are non-negative. For instance, let the dominant root be a given magnitude, hence $\mu_1$ positive and given, and $\mathbf{t}_1 = \mathbf{e}$. Now, $n^2 - 1$ parameters can be chosen, namely the elements of the vectors $\mathbf{t}_2, ...., \mathbf{t}_n$ and the remaining eigenvalues $\mu_2, ..., \mu_n$. $\mathbf{G} = \mathbf{T}^{-1}$ then is determined. However, even if the chosen coefficients are all positive, the resulting $\mathbf{A}$ will not necessarily be non-negative, since $\mathbf{G}$ may contain negative elements. Hence it is better to test the variability of the coefficients by starting from a given matrix $\mathbf{M}_1$ of rank one, to choose eigenvectors and a linear combination of the $\mathbf{M}_1$ with chosen small non-negative eigenvalues

To consider the coefficients of I/O tables as random numbers is unusual and seems not to have been done before in the Sraffa tradition, but it is clear that each coefficient $a_{ij}$ is subject to manifold accidental influences. Each is a statistical construct, based (in principle) on the observation of many firms in industry $\mathbf{a}_i$, which uses commodity $j$ under different circumstances (local variations of the weather, affecting different coefficients in different ways, local variations in the supply of commodity $j$ to the firms in industry $i$, working conditions). Marx therefore spoke of 'averages' and of 'socially necessary' techniques. The coefficients may vary and are up to a point uncertain in consequence. Dry years may mean more expense for water and less labour for the harvest. The multiplicity of the influences justifies the consideration of the coefficients $a_{ij}$ of each industry $i$ as independent. It could also be argued, however, that the coefficients in the columns are independent, since the commodities required by different industries depend on influences, some of which are industry-specific, like armaments on wars, building on money rates of interest etc. It remains to be seen whether a variant of the Goldstein-Neumann theorem can be proved, where the distributions on the columns of $\mathbf{A}$ are independent and identical on the rows.

To consider the coefficients on the rows as identically distributed seems inappropriate at first sight, since each industry appears to have a specific group of suppliers: the food

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so as to construct $\mathbf{A}$. Here is an example for $n = 2$: Let $dom\,\mathbf{A} = 1$ be given. We choose $m_{11} = 1/3$, hence $\mathbf{A}$ be written as

$$\mathbf{A} = \mu_1 \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix} + \mu_2 \mathbf{M}_2,$$

where $\mu_1 = 1$ is given. Next we choose $\mu_2 = 1/5$, so that the pre-assigned spectral gap is 80 %. We have $\mathbf{M}_2 = \mathbf{g}^2 \mathbf{t}_2$ and choose $t_{21} = 1$. Having chosen $n^2 - 1 = 3$ parameters, we obtain with $\mathbf{t}_1 = \mathbf{e}$ and $\mathbf{g}^1 = (1/3, 2/3)^T$ from the orthogonality relationships $\mathbf{t}_2 = (1, -1/2)$ and $\mathbf{g}^2 = (2/3, -2/3)^T$. One thus finds a modified

$$\mathbf{A} = \begin{bmatrix} 7/15 & 4/15 \\ 8/15 & 11/15 \end{bmatrix};$$

it is not obvious that this is a matrix with a small non-dominant eigenvalue.

I hope that this example helps to understand that matrices with small non-dominant eigenvalues illustrate the property of the random matrices of the Goldstein-Neumann theorem, for $\mathbf{A} = \mu_1 \mathbf{M}_1 + \mu_2 \mathbf{M}_2$ with $\mathbf{M}_1 = \mathbf{ce}$, $\mathbf{c} = \mathbf{g}^1$, and $\mu_2 \mathbf{M}_2$ representing the disturbance. The objection that input-output matrices do not look like matrices of rank one should thus be dispelled. Other objections like the reasons for abstracting from fixed capital have been discussed in the papers referred to.

industry depends on agricultural products, the steel industry has a specific supplier: coal. If industry $i$ delivers a unique input to industry $i+1$; $i=1,\ldots,n-1$; and industry $n$ delivers to industry 1, as in the following example for $n=3$

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & \alpha \\ \beta & 0 & 0 \\ 0 & \gamma & 0 \end{pmatrix},$$

or, more generally, if $a_{ij} > 0$ for $(i, j) = (1, n)$ and for $i = j+1$, $j = 1,\ldots,n-1$ and if $a_{ij} = 0$ otherwise, we have a circular system, $\mathbf{A}$ is imprimitive and the eigenvalues of $\mathbf{A}$ all have equal moduli; they are $n-th$ roots of $a_{1n}a_{21}\cdot\ldots\cdot a_{n,1-n}$. This is the extreme case of an industrial structure where each industry has one specific supplier. We discuss such matrices in the last note of section 7 below. Here, the non-dominant eigenvalues are not only not small, but they are all equal in modulus to the dominant root. However, as soon as we deviate from the circular pattern only a little, zero eigenvalues can appear. In the following example, where industry 1 produces for industries 2 and 3 and these produce for industry 1,

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

the eigenvalues are $\sqrt{2},\ -\sqrt{2},\ 0$.

The habit of economists to order industries according to the destination of the products (criticised by Sraffa in 1925, Sraffa 1925) causes them to imagine a nearly circular structure where the inputs in each industry form narrow groups of positive entries and the other $a_{ij}$ are supposed to be zero. This is less absurd at an extreme level of disaggregation, up to the point where each commodity produced and exchanged is an individual object, but the output of an industry consists of a class of goods – which usually are aggregated to one good – and the inputs are many; we then speak of single product systems.

The mistaken conclusion can be called the fallacy of mistaken arrangements. It is known that the throws of a die are i.i.d., even if the die is loaded. Let the numbers on the die be $0,1,\ldots,5$ instead of $1,\ldots,6$. Imagine that there are as many such dice as there are industries, and each die is loaded in a manner specific for the industry. Let $n$ such dice be thrown $n$ times. The matrix of the results is analogous to an input matrix $\mathbf{A}$. The random sequence in which the numbers $0$ to $5$ appear in row $1$ can be replaced by an ordered sequence by renumbering the columns in such a way that the first $a_{1j} = 0$, the next $a_{1j}$ equal one etc. up to the last $a_{1j} = 5$. Let the first industry represent food-processing, let the zeros correspond to no input, the fives to agricultural inputs, the other numbers to other inputs and let the first die be loaded so that there is a bias in favour of the fives: then we have an analogue for ordered arrangements of the inputs to the food-processing industry, which primarily needs foodstuff as inputs, and some others like lorries and buildings as well.

The fallacy now consists in the belief that the possibility of an ordered arrangement of the inputs in one industry contradicts the hypothesis of an identical distribution of the coefficients $a_{1j}$ of industry one and of the other industries. Identical means: almost each sample $\{a_{i,j1},...,a_{i,js}\}$ exhibits the same distribution for large $s$ and $n$; $1 \le j_1 < ... < j_s \le n$. The fallacy is double. First, the ordered arrangement is found only *ex post*; it is only one ordering among $n!$ permutations of the $a_{1j}$; identical distribution then means that the same distribution is generically encountered in almost all other permutations. Second, even if a clustering of inputs can be made to appear in an input-output table by suitable renumbering of the industries and commodities, this renumbering in one industry will immediately destroy any ordering that existed in the other industries. Otherwise, production would be nearly circular.

The intuition for the assumption, on which the paper is built, therefore is as follows: the output of each industry and commodity (or class of commodities) is one unit which represents total annual production of the commodity. Each $a_{ij}$ represents a share of this annual production, needed for reproduction. Each industry has a certain weight in the economy. Say, the car industry $i$ is large and represents 5 % of total output, measured as $\mathbf{a}_i\bar{\mathbf{p}} / \bar{\mathbf{p}}_i$ (in terms of the prices developed in the following sections). The $a_{1j}$ then must be roughly equal to 5 % on average. If $\mathbf{A} = \mathbf{ce} + \mathbf{P}$ as above, where $\mathbf{P}$ is a perturbation matrix with positive and negative entries and $\mathbf{A} \ge 0$, this means $c_i = 5$ %. If some commodity $j$ is not used in the car industry, $a_{ij} = 0$ and some other $a_{il}$ will be larger than 5 %. In this manner, 5 % is interpreted as the expected value of the coefficients $a_{1j}$ of industry $i$ by virtue of the weak law of large numbers. Even if the coefficients of empirical input-output matrices turn out only to approximately be independent and identically distributed, the hypothesis can legitimately be made as a strict assumption in the theory. Those who do not accept this as an explanation for the empirical distribution of the eigenvalues of input-output tables should propose another.

## 3. An Application to the Marxian Transformation Problem

Of course, there is much more to the Marxian theory of value than the transformation problem which has captured the attention of academic economists, but it is interesting not only because of the controversies surrounding it, but also because it is directly linked to the Marxian proposition that profits are nothing but redistributed surplus value. Since one such equality could trivially be established by choosing the surplus as the numéraire, a test of the proposition is whether the rates of profit measured in value terms and in price terms coincide and

$$r_{(value)} = \frac{M}{C+V} = \frac{P}{K+W} = r_{(prices)} \tag{1}$$

holds. $M$ here stands for surplus value (*Mehrwert*); otherwise, the notation is standard. The quantitative equality of surplus value $M$ and profits $P$ is based on the assumption, used by Marx himself, that the total product serves as numéraire, hence that $C+V+M = K+W+P$ by definition. $P = M$ then is not a trivial equality. It does, in fact, not hold in general, but only in special cases, but we want to argue that it holds in an important special case in a stochastic setting. Marx and Engels spoke of averages, with somewhat different interpretations of the term (see Schefold 2014). Since the interpretation has been discussed elsewhere, we here only focus on the quantitative relationship. To show that this exists is more than critics of Marx have admitted so far, but less than what Marx intended to show. It is one thing to demonstrate that the quantitative equality results, when it is derived from the input-output structure of the economy and another to insist that value is created by labour, that surplus value results from exploitation and that profits also qualitatively are surplus value, only distributed in proportion to capital advanced after having arisen in proportion to labour performed; the problem of the formal redundance of labour values remains.

Marx assumes that wages are advanced, and this assumption is used in Schefold (2014), but the proposition we are interested in follows also if wages are paid *ex post*, and we shall here adopt this convention so as to be able to use the same formulas afterwards for the analysis of the old neoclassical equilibrium.

Hence we have Sraffa prices

$$(1+r)\mathbf{Ap} + w\mathbf{l} = \mathbf{p} \tag{2}$$

in standard notation. Marx considers one technique at any one time, which evolves in the process of accumulation; it is, as already stated, not necessarily a dominant, but a socially necessary technique.

The vector of gross output $\mathbf{y}$, equal to the vector of activity levels, can be written as, if $\mathbf{b}$ is the vector of the commodities consumed by the workers and $\mathbf{s}$ the surplus in the Marxian sense, the vector of the commodities consumed by the capitalists in the stationary state (no net investment)

$$\mathbf{y} = \mathbf{yA} + \mathbf{b} + \mathbf{s}, \quad \mathbf{y} = (\mathbf{b} + \mathbf{s})(\mathbf{I} - \mathbf{A})^{-1}. \tag{3}$$

Profits $P = \mathbf{sp}(r)$, capital $K = \mathbf{yAp}(r)$ and wages $W = w\mathbf{yl}$ can be measured in the prices resulting from (2), as soon as a numéraire has been chosen, so that a wage curve $w = w(r)$ is given. The rate of profit can here be varied hypothetically, and the measurement is in prices proportional to labour values if $r = 0$, but there is an actual rate of profit $r^*$ which is consistent with the distribution of the commodities produced as expressed in (3). As one shows easily, there is exactly one actual rate of profit $r^*$ consistent with this physical distribution

$$r^* = \frac{P}{K + W} = \frac{\mathbf{sp}(r^*)}{\mathbf{yAp}(r^*) + w(r^*)\mathbf{yl}}. \tag{4}$$

The numéraire is denoted by $\mathbf{d}$; Marx in effect puts $\mathbf{d} = \mathbf{y}$. We now use the assumptions introduced in the first section. There are therefore $n$ linearly independent eigenvectors (rows) such that $\mathbf{q}_i\mathbf{A} = \mu_i\mathbf{q}_i$ and linearly independent eigenvectors $\mathbf{x}^i$ (columns) such that $\mathbf{Ax}^i = \mu_i\mathbf{x}^i; i = 1, ..., n;$ which allow to represent the numéraire vector $\mathbf{d}$ and the labour vector $\mathbf{l}$ as linear combinations: $\mathbf{d} = \sum \alpha_i\mathbf{q}_i$, $\mathbf{l} = \sum \beta_i\mathbf{x}^i$. It is possible to normalise the eigenvectors (so-called strong normalisation) such that

$$\mathbf{d} = \mathbf{q}_1 + .... + \mathbf{q}_n; \ \mathbf{l} = \mathbf{x}^1 + .... + \mathbf{x}^n, \tag{5}$$

as one proves easily; by adjusting the lengths of vectors $\mathbf{q}_i$ and $\mathbf{x}^i$, the $\alpha_i$ and $\beta_i$ all become equal to one. As we had ordered the eigenvalues according to their modulus, $\mu_1 = dom\,\mathbf{A}$ and $\mathbf{q}_1$ is proportional to the standard commodity; we call it the Sraffa vector. As is well known, the labour theory of value holds if and only if the labour vector is the right-hand side Frobenius eigenvector of the input matrix. Hence $\mathbf{x}^1$ is a positive vector which, if it was the labour vector, would lead to prices being equal to values; we call it the Marx vector. The deviations $\mathbf{m}$ of the numéraire vector from the Sraffa vector and the deviations $\mathbf{v}$ of the labour vector from the Marx vector are of interest:

$$\mathbf{m} = \mathbf{d} - \mathbf{q}_1 = \mathbf{q}_2 + .... + \mathbf{q}_n; \; \mathbf{v} = \mathbf{l} - \mathbf{x}^1 = \mathbf{x}^2 + ... + \mathbf{x}^n . \tag{6}$$

If $\mathbf{A}$ is random of type $\mathbf{ce}$, $\mathbf{e}$ tends to be the left-hand side eigenvector, for $\mathbf{eA}$ tends to $\mathbf{e}(\mathbf{ce}) = (\mathbf{ec})\mathbf{e}$; therefore $\mathbf{ec} = \mu_1$ and $\mathbf{e} = \mathbf{q}_1$. The standard commodity of such a system is proportional to the summation vector $\mathbf{e}$, but it is here *not* the numéraire and the economy is not in standard proportions, and yet we shall get $P = M$ !

This can be shown by means of the following formulas which are explained in more detail in the paper referred to. Using the abbreviation $\rho = 1 + r$ and the formula $(\mathbf{I} - \rho\mathbf{A})\mathbf{x}^i = (1 - \rho\mu_i)\mathbf{x}^i$, we get a representation of prices

$$\mathbf{p} = w(\mathbf{I} - (1+r)\mathbf{A})^{-1}\mathbf{l} = w\sum_{i=1}^{n} \frac{\mathbf{x}^i}{1 - \rho\mu_i} \tag{7}$$

which is a quite general formula of prices; prices in terms of the wage rate $p / w$ are a sum of hyperbolas. If the spectral gap is large enough (if the conditions of the theorem by Goldstein and Neumann are fulfilled), prices can be approximated by setting $\mu_2 = ... = \mu_n = 0$ and (7) is transformed into

$$\mathbf{p} = w\left[\frac{\mathbf{x}^1}{1 - \rho\mu_1} + \mathbf{v}\right]. \tag{8}$$

We now use that gross outputs or activity levels $\mathbf{y}$ are the numéraire $\mathbf{d}$ and the orthogonality $\mathbf{q}_i\mathbf{x}^j = 0$, $i \neq j$, to get

$$1 = \mathbf{yp} = w\left[\frac{\mathbf{q}_1\mathbf{x}^1}{1 - \rho\mu_1} + \sum_{i,j=2}^{n} \mathbf{q}_i\mathbf{x}^j\right] = w\left[\frac{\mathbf{q}_1\mathbf{x}^1}{1 - \rho\mu_1} + \mathbf{mv}\right]; \tag{9}$$

the wage curve is a hyperbola. We have so far used only the standard assumptions for Sraffa systems and the assumptions for a large spectral gap. We can invert our proposition and say: *Wage curves of Sraffa systems are simple hyperbolas except for wiggles due to non-dominant eigenvalues which are not zero.*

But here we focus on random systems which allow to go a step further. We consider the components of the deviation vectors $\mathbf{m}$ and $\mathbf{v}$ as random variables which we may assume to be uncorrelated, for the composition of output depends on factors such as the taste of consumers and the labour vector represents technology. Making therefore our second assumption $\operatorname{cov}(\mathbf{m}, \mathbf{v}) = 0$, one gets from the standard formula for the covariance $\mathbf{mv} = n\bar{m}\bar{v}$, where $\bar{m}$ and $\bar{v}$ are averages $\bar{m} = \mathbf{em}/n$ and $\bar{v} = \mathbf{ev}/n$. But the summation vector $\mathbf{e}$ happens to be, in the limit, the Frobenius eigenvector of $\mathbf{A}$ and we can use the orthogonality relationships

$$n\bar{\mathbf{v}} = n\mathbf{ev}/n = \mathbf{e}(\mathbf{l} - \mathbf{x}^1) = \mathbf{e}(\mathbf{x}^2 + ... + \mathbf{x}^n) \to 0 . \tag{10}$$

It follows from this second assumption (about the covariance) that the expression $\mathbf{mv}$ in (9) can be replaced by this result, written as $n\bar{m}\bar{v} = 0$, so that (9) yields a *linear wage curve*

$$\bar{w} = \frac{1 - \rho\mu_1}{\mathbf{q}_1\mathbf{x}^1} ; \tag{11}$$

we can insert it into the price equations (8) and get

$$\bar{\mathbf{p}} = \frac{\mathbf{x}^1}{\mathbf{q}_1\mathbf{x}^1} + (1 - \rho\mu_1)\frac{\mathbf{v}}{\mathbf{q}_1\mathbf{x}^1} ; \tag{12}$$

the prices and the wage rate in (12) and in (11), $\bar{\mathbf{p}}$ and $\bar{w}$, are now expressed in terms of the numéraire. We have therefore obtained a *linear wage curve*, although we have neither used the standard commodity nor the condition that prices are equal to values. *Prices are here a linear function of the rate of profit*, while the price vectors at $n$ different rates of profit are linearly independent in the general case. A third result is the following. We can speak of an average of prices or values, given the normalisation. The average of prices is $\mathbf{e}\bar{\mathbf{p}}/n$. Since $\mathbf{ev} = 0$, the average of prices is independent of the rate of profit:

$$(1/n)\mathbf{e}\bar{\mathbf{p}}(r) = (1/n)\frac{\mathbf{ex}^1}{\mathbf{q}_1\mathbf{x}^1} = 1/n ; \tag{13}$$

this means that, for a given system and given $n$, *prices and values are equal on average*. That the average price or value must tend to zero, if normalised and for $n \to \infty$, is obvious. The Marxian proposition that aggregates measured in prices and values must be the same on average in the economy at large has here found a precise theoretical expression, as a proposition which is true in the limit. But the individual prices are not equal to values.

However, a third assumption is required to apply the statement to the assertion about profits and surplus value. The physical surplus is $\mathbf{s}$. We assume, for analogous reasons as above, that $\operatorname{cov}(\mathbf{s}, \mathbf{v}) = 0$. Total profits are equal to

$$P = \mathbf{s}\bar{\mathbf{p}} = \frac{\mathbf{sx}^1}{\mathbf{q}_1\mathbf{x}^1} = \frac{\mathbf{sx}^1}{\mathbf{ex}^1}, \qquad (14)$$

because $\mathbf{sv} = n\overline{sv}$ and $n\bar{v} = 0$. *The amount to profits* remains constant with a virtual variation of the rate of profit and *is* therefore *equal to* the value of the surplus or *surplus value,* as Marx postulated. To interpret formula (14), one can imagine a modification of the actual system in which the actual labour vector is replaced by the Marx vector. Prices are proportional to labour values in this modified economy and profit in the actual economy is equal to profit in the modified economy, where the labour theory of value holds, as formula (14) shows; the values are normalised by dividing by the sum of the components of the Marx vector. The main result, however, is that *the rate of profit in value terms and in price terms* coincide, hence that (1) is fulfilled.

## 4. A general equilibrium model according to the old neoclassical theory with the value of capital K given

Modern general equilibrium theory (intertemporal) is marred by its main success. The existence of equilibria is proved under conditions which are so general that we do not really know how the solutions look like, in particular, whether they are stable and unique or how the time path of the solutions evolves over time. Another important problem is caused by degenerate solutions where one of the distributional variables is zero. We do not worry if the rent of desert land vanishes, but what does equilibrium mean if the wage is zero? It is no wonder that economists often turn to the production function, if they want to derive definite results for the theory of growth, but the production function implies that one deals with long-period positions, be it in the form of comparisons or a slow transformation of the data as in Solovian growth models. It is implicitly assumed that the solutions are normal in that there is a unique rate of interest. Full employment follows from marginal productivity, with clear exceptions due to limited possibilities of substitution or the imposition of disequilibrium prices by imperfect competition or state intervention. The conditions under which the production function works can be clarified by formulating the model of the old neoclassical equilibrium explicitly. It will now be shown that the conditions for its functioning are basically the same as those required for the existence of an approximate surrogate production function, and these in turn are similar to the conditions which we encountered in our discussion of Marx, with complications mainly due to the problem of representing technical substitution in a general equilibrium framework. We thus concentrate on the conditions for the existence of normal solutions.

We assume constant returns to scale, for the reasons advanced by Sraffa in 1925 which are here appropriate, and I propose to interpret the spectrum of techniques as in my papers on the surrogate production function referred to above, where the techniques are those represented by input-output tables with $n$ sectors and the number of tables corresponds to the number of countries $h$. Even the most enthusiastic defender of liberalism cannot postulate that entrepreneurs are omniscient. I like to think in this stylized model that entrepreneurs in $h$ countries, say $h=10$, know the techniques employed in their sector (the techniques in the industries in which they are active themselves) and that they have some knowledge of what their rivals are doing in the other countries, while they have only vague ideas of what happens in other sectors than their own. The knowledge about technology thus is decentralised. The level of aggregation is assumed to be such that there are no significant links between the sectors so that, in principle, each of $h$ methods employed in one sector can be combined with any of $h$ methods employed in any other sector. It is therefore possible to select $h^n$ combinations of methods from the $h$ input-output tables. The number of combinations is, for instance, $10^{100}$, if $h=10$ and $n=100$. In all $h$ economies, entrepreneurs strive to find and to employ the best method. The theory normally deals with the ideal solution, which is, given the rate profit, the cost minimising technique; it is then the same for all countries. But reality never quite achieves this, and success is different in different countries and industries. Hence we assume that competition has resulted in different techniques in different countries. We can leave it open whether the input-output tables in different countries reflect a different average for each industry or whether only one method is used uniformly in each industry in each country. This is how I like to think about the matter in order to get a satisfactory representation of the theory in a field where realism is very difficult to approximate and realism often is claimed for quite daring intellectual constructions. The assignments of methods to countries serve as an illustration; it is not necessary for what follows. The formal analysis remains the same if one makes more general assumptions and postulates a large finite book of (as in part yet unrealised) blueprints for methods of production. By contrast, the systematic invention of new techniques, the production and possibly also the use of newly invented machines under conditions of imperfect competition and/or increasing returns to scale as in modern theories of endogenous growth do not fit in here.

But the formal results are based on more conventional assumptions. For what follows, it suffices to assume that there are $s$ techniques, where $s=h^n$ in the illustration just discussed. The techniques $\sigma$; $\sigma=1,...,s$; are denoted $\mathbf{A}^\sigma, \mathbf{l}^\sigma$. For purposes of general equilibrium theory, it is convenient to arrange all methods of production in one rectangular matrix, of $hn$ rows and $n$ columns, if we stick to the illustration. $\mathbf{A}$ then consists of a column of $h$ input-output tables. It is not excluded that some countries use the same method for the production of a particular commodity. The matrix is associated with a corresponding labour vector and an output matrix which repeats the unit matrix $\mathbf{I}$ $h$ times, arranged in a column to form output matrix $\mathbf{B}$:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^1 \\ \vdots \\ \mathbf{A}^h \end{bmatrix}, \quad \mathbf{l} = \begin{bmatrix} \mathbf{l}^1 \\ \vdots \\ \mathbf{l}^n \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{I} \\ \vdots \\ \mathbf{I} \end{bmatrix}$$

$\mathbf{A}^i, \mathbf{l}^i$ are therefore the input-output tables with associated labour vectors; $i = 1, ..., h;$ and $\mathbf{A}^\sigma, \mathbf{l}^\sigma$ the techniques which can be formed by combining methods from these tables. The activity level vector is $\mathbf{y}$, with $hn$ components, if we refer to the spectrum of techniques $(\mathbf{A}, \mathbf{l})$, and we write $\mathbf{y}^\sigma$, with $n$ components, when we speak of gross outputs and the activities of any given – possibly cost minimising – technique $\sigma$.

There are $n$ commodities which are both capital goods and consumption goods (for a similar model where consumption goods and capital goods are separate, see the Walras-Morishima model discussed in Schefold (2015)). These commodities must be available in definite proportion to guarantee the stationary reproduction which we want to represent; the appropriate composition which must be available as the stocks for production is denoted by $\mathbf{f}$. The composition of $\mathbf{f}$ remains to be determined. There is labour of one kind, available in quantity $L$, and there is, apart from the capital goods $\mathbf{f}$, also capital as the value magnitude $K$. John Bates Clark (1899) insisted that capital $K$ is mobile, while capital goods are not. The modern interpretation of this is to say that capital is “malleable”, but Clark thought that the capital goods, with total value $K$, could individually be sold and replaced by other goods. It may sound shocking, but this conception is not so different from the Marxian one when he spoke of the circulation of capital: in the Second Volume capital is advanced in monetary form, this money capital changes form, in that the capital is used by means of production and labour power, production takes place, value is added, the produced commodities are sold and surplus value is realised, if total production is purchased. The proceeds in the form of money are distributed as revenue, but in part turned into money capital for reproduction. The reproduction of this stock of value is precarious in Marx because the realisation may fail, but Clark was confident that capital could be preserved in the abstract, even when it was a matter of changing techniques in the face of obsolescence. One example he gives is the switch from the catch of whales to the manufacturing of cloth: “As the vessels (of the whale catchers – BS) were worn out, the part of their earnings that might have been used to build more vessels was actually used to build mills. The nautical *form* (! BS) of the capital perished; but the capital survived and, as it were, migrated from the one set of material bodies to the other” (Clark 1965 [1899], p.118).

The idea that capital is physically malleable is, of course, a metaphor. There only can be realism in the idea of a lasting stock of capital as value, taking the form of ever changing capital goods, if one pursues these transformations and describes how capital goods are bought and sold, ever changing the form, while the substance remains the same – not with the invariability of a natural law such as that of the preservation of energy or mass, but with precariousness encountered at all stages of the process, in particular at the stage of “realisation”.

Equilibrium equations, some of them here reproduced in a modernised form as inequalities, do not capture the precarious (not utopian!) nature of the process of circulation. Who will buy all the horses, when the farmers switch to tractors? But the transition is possible gradually if, as in Clark’s example, amortisation funds or the replacing of old horses are used for a gradual modernisation. The argument of asset-specificity should neither be dismissed nor exaggerated.

We arrive at the following equations, partly modernised as inequalities, for the old equilibrium. Given is the technology $(\mathbf{A}, \mathbf{B}, \mathbf{l})$. The techniques to be used are endogenous (unknown). Given are also the amount of capital in value terms, $K$, and the labour supply $L$; distribution, prices, activity levels, consumption demand and the physical composition of capital $\mathbf{f}$ are endogenous (unknowns):

$$(1+r)\mathbf{Ap} + w\mathbf{l} \geqq \mathbf{Bp} \qquad (15.1)$$

$$\mathbf{y}\left[(1+r)\mathbf{Ap} + w\mathbf{l} - \mathbf{Bp}\right] = 0 \qquad (15.2)$$

$$\mathbf{yA} = \mathbf{f} \qquad (15.3)$$

$$\mathbf{yl} \leqq L \qquad (15.4)$$

$$\left[L - \mathbf{yl}\right]w = 0 \qquad (15.5)$$

$$\mathbf{fp} \leqq K \qquad (15.6)$$

$$\left[K - \mathbf{fp}\right]r = 0 \qquad (15.7)$$

15.4 and 15.6 could be written as equations, if we had an infinite spectrum of techniques, as we shall see. The mixture of inequalities and equalities will compel us also to use a ‘mixed’ kind of proof for the existence, and, in the subsequent section, stability of equilibrium. The equations mean successively that prices are competitive, that only processes covering costs are activated, that production fully uses the available capital goods, that employment is bounded by available labour, that unemployment leads to a zero wage, that the value of the capital goods employed does not exceed the value of capital available and that the rate of interest is zero, if the value of capital is not fully employed. Demand for consumption goods results from the maximisation of utility. Since we are considering a system in stationary reproduction, we do not consider net savings (growth will be considered in a future paper) and saving for reproduction is implicit. Hence we have a vector of demand for consumption goods, which depends on prices and distribution. Demand (15.8) is positive and continuous by assumption, if there is any income at all. Walras' law (15.9) expresses that net income is spent on consumption.

$$\mathbf{z} = \mathbf{z}(\mathbf{p}, w, r) \qquad (15.8)$$

$$rK + wL = \mathbf{zp} \qquad (15.9)$$

Capital must be reproduced. Gross investment is denoted as vector $\mathbf{f}^*$. What determines this investment? Walras assumed an elastic supply of capital goods, which were used to satisfy consumption (15.10) and to produce new capital goods – investment – where there was capacity left, given savings (Schefold 2015). Keynesians have investment determined by expectations. At first sight, our formulation seems closer to Walras. $\mathbf{f}^*$ cannot exceed total production minus what is taken away for consumption (15.10) and stationarity requires that the composition of capital remains the same (15.11). We postulate equalities:

$$\mathbf{z} + \mathbf{f}^* = \mathbf{yB} \qquad (15.10)$$

$$\mathbf{f}^* = \mathbf{f} \qquad (15.11)$$

This last of our 11 equations is formally the simplest, but conceptually perhaps the most difficult. All others result directly from the theory of demand and supply, broadly speaking: if demand equals supply, transactions take place at prices equal to normal costs. If demand falls short of supply, prices fall. It is not so obvious how the adjustment takes place if demand exceeds supply, since prices exceeding costs are simply ruled out by implicit reference to competition. It is, however, the advantage of the old neoclassical model that normal solutions can be constructed, and their stability properties are relatively transparent. This holds, by and large, except for equation (15.11) which formulates the condition for a stationary long-period position. It should be established by gravitation. It looks like an investment function for static entrepreneurs without the ability or the will to grow and to innovate as in a Schumpeterian stationary state. The equation embodies an idea that is common to classical and neoclassical economics: in the long run, the economy is stationary in the absence of external shocks like harvest fluctuations; it does not grow in the absence of innovations (no new goods, now new methods of production) and with a stationary population. Effective demand (needs) and distribution would have to be fixed in the classical case. Here, in neoclassical equilibrium, work effort is just compensated by the utility of consumption and the drive to accumulate (saving) is offset by the desire to enjoy present gratifications. The ideal of stationarity, absent the causes for its disturbance, thus can be explained: the given needs or preferences guide entrepreneurs *indirectly* to invest neither more nor less than is necessary for replacement.

We proceed to solve equations (15). Let, to begin with, a level of distribution be given – it will be determined last to close the system. Normal prices result, given distribution in the form of a rate of interest $r\ (0 \le r \le R)$, from cost minimisation and the choice of a numéraire $\mathbf{d}$. We have, except at a switchpoint which we may at first ignore, a unique cost minimising technique $\sigma$ and a wage curve $\overline{w}_\sigma(r)$ and prices $\overline{\mathbf{p}}_\sigma(r)$ as above (equations 11 and 12), if the same assumptions as in the Marxian case are made regarding the techniques ($\mathbf{A}^\sigma \ge 0$ productive, indecomposable, diagonalisable, $\mathbf{l}^\sigma > 0$). It may seem a more drastic assumption (to be introduced later) to demand that the matrices be random and that the covariance conditions be fulfilled, if there is a large spectrum of techniques, but in the Marxian case the assumption was made for any technique, which really amounts to the same. To each technique there is, according to the corresponding prices, a consumption demand $\mathbf{z}_\sigma(r)$ which we assume to be positive as soon as $(\overline{w}, r) \ge 0$. The consumption vector demanded according to the prices at the rate of interest $r$ is denoted by $\mathbf{z}(r)$; we have $\mathbf{z}(r) = \mathbf{z}_\sigma(r)$ as long as we are not at a switchpoint. Since we are in stationary reproduction, we must have, with $\mathbf{y}_\sigma$ as the vector of activity levels for the technique chosen, $\mathbf{y}_\sigma = \mathbf{z} + \mathbf{y}_\sigma \mathbf{A}^\sigma$; the activity levels, given distribution, hence the technique and consumption, are therefore determined by

$$\mathbf{y}_\sigma = \mathbf{z}(\mathbf{I} - \mathbf{A}^\sigma)^{-1} > 0\,. \tag{16}$$

The point is that $\mathbf{z}_\sigma$ is determined, once prices are given, and since prices depend only on the rate of interest, $\mathbf{z}_\sigma$ is a function of $r$ and so is $\mathbf{y}_\sigma$. Now let a vector $\mathbf{f}_\sigma$ be defined by $\mathbf{f}_\sigma = \mathbf{y}_\sigma \mathbf{A}^\sigma$. The augmented activity level vector $\mathbf{y}$ has the same components as $\mathbf{y}_\sigma$, where applied to the corresponding activities, and zeros otherwise. With this, equations (15.1), (15.2) and (15.3) are fulfilled. One thus obtains the unknowns $\mathbf{f}_\sigma = \mathbf{y}_\sigma \mathbf{A}^\sigma$ and $\mathbf{f}^*_\sigma = \mathbf{f}_\sigma$, fulfilling (15.3), and (15.11) for the corresponding augmented vector $\mathbf{y}$. From (16) and our definitions, we get

$$\mathbf{y}_\sigma - \mathbf{y}_\sigma \mathbf{A}^\sigma = \mathbf{y}_\sigma - \mathbf{f}_\sigma = \mathbf{y}_\sigma - \mathbf{f}^*_\sigma = \mathbf{z}$$

hence

$$\mathbf{z} + \mathbf{f}^*_\sigma = \mathbf{y}_\sigma = \mathbf{y}_\sigma \mathbf{B}_\sigma,$$

where $\mathbf{B}_\sigma$ is the output matrix pertaining to technique $\sigma$. This output matrix is for each technique a unit matrix. For the augmented vectors and matrices we get (15.10). To be precise: we can write $\mathbf{f} = \mathbf{f}_\sigma$, $\mathbf{f}^* = \mathbf{f}^*_\sigma$ and $\mathbf{z} = \mathbf{z}_\sigma$ for the quantity vectors of dimension $n$, but not so for the activity levels and matrices: $\mathbf{y}_\sigma, \mathbf{A}_\sigma, \mathbf{B}_\sigma$ consist of the $n$ elements and rows of $\mathbf{y}, \mathbf{A}, \mathbf{B}$ which are positive or activated respectively. The asymmetry between prices (unprofitable activities may well exist but are not used) and quantities (no overproduction appears) arises because we have single production and all goods are both, capital goods and consumption goods.

(15.9), Walras' law, is true identically. We still need to determine (15.4), (15.5), (15.6) and (15.7). From (2) in the form $(\mathbf{I} - \mathbf{A}^\sigma)\bar{\mathbf{p}}_\sigma = r\mathbf{A}^\sigma\bar{\mathbf{p}} + \bar{w}_\sigma \mathbf{l}^\sigma$ and using (16), we get Say's law:

$$rK + w_\sigma L = \mathbf{z}_\sigma \bar{\mathbf{p}}_\sigma = \mathbf{y}_\sigma (\mathbf{I} - \mathbf{A}^\sigma)\bar{\mathbf{p}}_\sigma = r\mathbf{y}_\sigma \mathbf{A}^\sigma \bar{\mathbf{p}}_\sigma + \bar{w}\mathbf{y}_\sigma \mathbf{l}^\sigma = r\mathbf{y}\mathbf{A}\bar{\mathbf{p}} + \bar{w}\mathbf{y}\mathbf{l}. \qquad (17)$$

the revenues on the left and on the right are adequate to buy the output for consumption in the middle. The supply of capital and labour is on the left, the demand at each level of $r$ on the right. It follows that equilibrium on one market, for instance the labour market with $\mathbf{y}_\sigma \mathbf{l}^\sigma = L$ (equations 15.4 and 15.5), implies equilibrium on the aggregate capital market (equations 15.6 and 15.7, using 15.3). Hence the famous idea that one variable which has still not been determined, distribution, so far given in the form of an arbitrary rate of interest, can be varied to clear one of these markets in order to also have cleared the other. The task looks deceptively simple: fixing the wage rate to clear the labour market seems to be the obvious solution. Petri (2004) confirms that this was the approach of the old neoclassicals.

The argument, modernised, is as follows. We have a finite spectrum of techniques. There is a largest maximum rate of profit among all maximum rates of profit, say of a technique $\tau$, denoted $R_\tau$, and the rate of profit, also called the rate of interest, varies between zero and this maximum rate. The wage varies accordingly and in a strictly monotonic fashion between its maximum at $r = 0$ and zero at $R_\tau$. This is because prices change continuously at each switch of techniques. Hence $\mathbf{z}(r)$ is continuous – consumption demand is defined uniquely on the envelope at $r$ – but $\mathbf{y}_\sigma(r)$ and $\mathbf{f}$, activity levels and capital composition change discontinuously at switchpoints according to (16), where there is a transition between two techniques; generically in the form that all methods of production in all industries but one remain the same, and in this one industry the switch takes place from one method of production to another: in such a way that any convex linear combination of the two methods is also feasible. We make the generic assumption that only two solutions $\mathbf{A}^\sigma$ and $\mathbf{A}^\vartheta$ are eligible at the switchpoint, with their convex combinations. It follows that $\mathbf{y}(r)$ and hence labour demand $\mathbf{yl}$ is an upper semi-continuous correspondence, dependent on $r$. Labour demand is a vertical segment at switchpoints and otherwise rising or falling; it can be visualized as a continuous curve with corners in the plane, but it is not a function because of the vertical segments. Note that consumption demand and therefore activity levels and the composition of capital may change with distribution, even if relative prices do not change for a particular technique and even if the corresponding wage curve is linear, for the composition of demand depends on distribution, if the tastes of capitalists and workers differ.[5] Hence we expect vertical segments in the demand correspondence, and intervals where labour demand rises or falls, but horizontal segments are exceptional.[6]

Labour demand $L^D(r) = \mathbf{yl}$ will intersect the curve for the labour supply $L^S = L$ (which in our case is a constant) at one (case a) or at several (case b) rates of profit. Or $L^D \geq L^S$ for all $0 \leq r \leq R_\tau$ (but $L^D > L^S$ for all $r$ turns out not to be possible – capital is in excess supply and labour in balance – case c). Or supply exceeds or is at most equal to demand $(L^D \leq L^S)$ everywhere (case d). The intersection of the labour demand correspondence with the given supply determines the technique, if we are not at a switchpoint, or, if we are at a switchpoint, a convex linear combination of two techniques, which will clear the labour market. Thus $\lambda$, $0 < \lambda < 1$, will be found such that $\lambda \mathbf{y}_\sigma \mathbf{l}^\sigma + (1-\lambda)\mathbf{y}_\vartheta \mathbf{l}^\vartheta = L$, $\sigma$ and $\vartheta$ being the coexisting techniques at the switchpoint. The superposition concerns the industry, in which the change of method takes place. It follows from (17), at a switchpoint where the superposition occurs, or if there is no switch, that the capital market will also be in equilibrium in cases (a) and (b), provided that $w$ and $r$ are positive. If there is no $r$ in $0 < r < R_\tau$ with $\mathbf{yl} = L$, we have either $\mathbf{yl} > L$ or $\mathbf{yl} < L$ in $0 < r < R_\tau$, therefore, because of the upper semi-continuity of the correspondence, either case (c) or case (d). In case (c), an excess of the labour demand over the labour supply cannot hold for all rates of profit, $0 \leq r \leq R$. For try $r = 0$, hence $w > 0$, using the price equations for the system or the linear combination of two systems on the envelope at $r = 0$, if there is a switchpoint at $r = 0$. (17) then yields (15.5) and, dividing by $w > 0$, 15.4, with equality. There is $\bar{r} > 0$ and an open interval $J = \{0 < r < \bar{r}\}$ such that there is no switchpoint in $J$. By (17) and by the assumption of case (c), we have $K \geq \mathbf{yAp}$ in $J$ and, since the correspondence is upper semi-continuous, also in $r = 0$. So, using

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[5] For instance: If $w$ is small, workers have a strong preference for bread, if $w$ is higher, they prefer leisure goods produced at low levels of industrial activity, thus making room for goods to be produced for the capitalists. In this sense the supply of labour is elastic despite the given $L$ and the neoclassical theory of distribution comes to the surface.

[6] The important exception is the one-good model with only one method of production, hence with $\mathbf{A} = a$, $\mathbf{l} = l$; $a, l$ scalars. The reader can veryfy by calculation that one gets an indifferent equilibrium for $0 < r < R$, if by chance $K / L = a / l$. If $K / L > a / l$, the equilibrium is at $r = 0$, if $K / L < a / l$, it is at $r = R$. Cf. also (18) below.

(15.3), (15.6) is fulfilled and the equilibrium is at $r = 0$;[7] in particular the labour market must be in equilibrium, with equalities in (15.4). Conversely in case (d): if the labour supply exceeds labour demand, 15.4 holds with inequalities, the wage rate is zero and (17) implies equilibrium on the capital market at $r = R_\tau$ (15.6 and 15.7).[8]

*But these extreme solutions* (c) *and* (d), which we have constructed, *are degenerate!* The wage in particular should not only not be zero but reach a subsistence minimum. Debreu (1959) ensured a minimum standard of life by assuming that each consumer had enough resources to live by his own means, if necessary. We want to find a necessary and sufficient condition, which ensures normal solutions, and this will lead us to the conditions required for the construction of the production function. The following proposition is crucial (for simplicity, it is formulated for solutions with a unique cost-minimizing technique):

*A solution to equations* (15) *will be a normal solution with equilibrium both in the capital and in the labour market and with* $w > 0$ , $r > 0$ , *if and only if*

$$\frac{K}{L} = \frac{\mathbf{y}_\sigma \mathbf{A}^\sigma \overline{\mathbf{p}}_\sigma}{\mathbf{y}_\sigma \mathbf{l}^\sigma}\,, \tag{18}$$

*where* $\sigma$ *denotes the cost minimising technique at the level of distribution given by the solution.* We denote $\mathbf{y}_\sigma \mathbf{A}^\sigma \overline{\mathbf{p}}_\sigma / \mathbf{y}_\sigma \mathbf{l}^\sigma = k$. Condition (18) obviously is necessary: it follows from (15.8), (15.6) and (15.5), if $w > 0, r > 0$. It is also sufficient. For if the supply of capital is not equal to the demand, without loss of generality exceeding it, it follows from (18) that $K / (\mathbf{y}_\sigma \mathbf{A}^\sigma \overline{\mathbf{p}}_\sigma) = L / (\mathbf{y}_\sigma \mathbf{l}^\sigma) > 1$, but (17) implies, if both $w > 0$ and $r > 0$, that an excess supply in one market means an excess demand in the other. Hence we have a contradiction; demand and supply must be equal, if $w > 0$ and $r > 0$. Note that this condition must be expressed and hold for the appropriate linear convex combination of two coexisting techniques at a switchpoint, as determined by an intersection of the labour demand correspondence with the supply.

If we admit degenerate solutions, there will always be an equilibrium, given our assumptions, but it follows from the proposition expressed by (18) that *a normal solution exists if and only if the labour demand correspondence cuts the labour supply function between zero and the maximum of the maximum rates of profit.*

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[7] Case (c) illustrustrates how supply limits demand in the neoclassical models. Keynesians suppose that credit intervenes to lift demand to a higher level. The expansion is modelled as a multiplier process and the new equilibrium can be sustained, once expectations have lead there.

[8] We could dispense with a fixed point theorem in this proof, because we used the theory of normal prices and were dealing with two factors only.

## 5. Stability

We are now interested in the macroeconomic stability properties of normal solutions. We begin with an intuitive consideration. The equilibrium solution is not stable, if the optimal technique $\sigma$ at each given $r^*$ is, in the neighbourhood of equilibrium, not such that $\mathbf{y}_\sigma \mathbf{A}^\sigma \underline{\mathbf{p}}_\sigma / \mathbf{y}_\sigma \mathbf{l}^\sigma = k(r)$, the intensity of capital, falls as $r$ rises. This will be true even if there is only one technique and the intensity of capital changes because of Wicksell effects. For if the amount of capital is given and the rate of profit rises and the wage rate falls, a rise of the capital intensity such that $k(r) > K/L$ for $r$, rising above $r^*$, would mean that the demand for labour would fall so that, given an initial disturbance in the labour market, unemployment would be increasing, causing the wage to fall further and to move away from the equilibrium value. Such an instability would also follow from reswitching and reverse capital deepening. The consideration rests on a simplification. We only look at the markets for capital and labour, without analysing how the deviation of the distributive variables from equilibrium might lead to disequilibria in the other markets. In other words: we are only interested in the macro stability problems characteristic for capital theory. 'Micro' stability problems do occur in this model, but they will have to be singled out on another occasion.

The key to the consideration is the inverse relationship of the intensity of capital and of the rate of profit. As the rate of profit rises from zero to $R_\tau$, the intensity of capital should fall in principle from infinity to zero, since the given ratio $K/L$ can *a priori* be anything. This is why the Inada conditions are postulated for production functions. If they are fulfilled, a normal equilibrium is assured. But the spectrum of techniques is inherently finite in our case. It can only approximate the extremes of the capital-labour ratios, in that there is always a finite positive minimum and maximum for $k$. Hence we have condition (18) for normal solutions, which is not only sufficient, but also necessary. In order to obtain normal solutions for all $K/L$, one needs an infinite spectrum. As long as the assumption is not made, to avoid degenerate solutions and to fulfil (18), $K$ and $L$ must be assumed to lie within certain bounds which depend on the technology. And it must now be made clear how the individual techniques have to be characterised so that a monotonic fall of the capital-labour ratios can be observed on the envelope of the wage curves, so as to avoid the instabilities due to capital reversals due to Wicksell effects, reswitching and reverse capital deepening.

Given the numéraire, all wage curves are defined within the bounds of normal solutions, and those of cost-minimising techniques will be situated on the envelope. It seems at first sight that the capital-labour ratios cannot properly be compared between techniques, since these depend not only on the technique and the level of the rate of profit, but also on the composition of output that varies from technique to technique according to relative prices, because of the choice of consumers (only the numéraire remains the same). However, capital labour ratios have to be compared between techniques only at switchpoints, and there relative prices and distribution coincide for both techniques involved in the switch, hence the net product will be the same for both techniques in the stationary state. This common net product may now be chosen as the numéraire in order to analyse stability locally at the switchpoint, using the familiar duality relationships derived from the wage curves based on this numéraire. The intensity of capital can be read off the wage curve, using $y = w + rk$, hence $k(r) = (1/r)\left(y(r) - w(r)\right) = (1/r)\left(w(0) - w(r)\right)$; it follows that $k(r)$ falls at the switchpoint as $r$ rises if and only if there is neither reswitching nor reverse capital deepening. In-between switchpoints, only one wage curve dominates, and $k(r)$ will fall as $r$ rises if and only if anti-neoclassical (perverse – see section 6) Wicksell effects are absent, always taking the net product demanded at any $r$ as the numéraire in order to analyse stability at $r$ by considering the wage curve at $r$. This analysis, based on the choice of a local numéraire, is unambiguous at switchpoints, because the wage curves which dominate at any $r$ or which are involved in a switchpoint do not depend on the numéraire, only the shape of the wage curves depends on it. Wicksell effects are numéraire-dependent. The instabilities then are important, if the numéraire has relevant economic meaning.

The composition of output changes with distribution in the old neoclassical equilibrium. The formula $k(r) = (1/r)\left(y(r) - w(r)\right) = (1/r)\left(w(0) - w(r)\right)$ presupposes that net output is taken as the numéraire, since it ensures that $y(r) = w(0)$: net output as the numéraire is constant and equal to the wage, where there are no profits. Hence we have been able to use the formula so far only for a local stability analysis, taking the local net output as the numéraire. Assume now that net output does not change with distribution and is taken as the numéraire. Different techniques are then used to produce the same composite good and can be substituted to this end according to the level of factor prices as in neoclassical theory. If the wage curves are sufficiently numerous to approximate a continuum, the slope of the envelope of the wage curves is equal to the slope of the individual linear wage curve tangent to it so that $k(r) = -\hat{w}'(r)$, where $\hat{w}$ is the envelope. Schefold (2013a) discusses the problem, which arises if wage curves are not exactly straight, so that this calculation of the intensity of capital by means of the derivative and that by means of the formula $(y - w)/r = k$ differ, $y$ being the output per head of the individual technique chosen at $r$. Two different, contradictory measures for output per head then are obtained; I have called this difference declination. It is certain to disappear only if wage curves are straight. The assumption of nearly straight wage curves therefore is sufficient, but not generally necessary for the existence of a stable neoclassical equilibrium. However, the assumption of straight wage curves is convenient if one wants to be sure to avoid declination and the instabilities due to anti-neoclassical (perverse) Wicksell effects, reswitching and reverse capital deepening.

Having gone so far, we can extend the comparison with the construction of the surrogate production function by considering what follows, if the net product does not change with distribution, but is given. All techniques produce the same output at all levels of distribution in the stationary state.[9] The techniques must not only be infinite in number but also such that for every $K/L$ given, there is a linear wage curve of slope $w' = k = K/L$. This will be approximately the case, if the wage curves derive from random systems, are sufficiently numerous and evenly spaced so that switchpoints are close to each other. However, as $w/r$ rises, there must be techniques with output per head large enough and the maximum rate of profit small enough to approximate any large $K/L$, and conversely, if $w/r$ falls. In other words: output per head must go to infinity, as $r$ falls, and the maximum rates of profit rise without bound, as $r$ rises.

[9] Zambelli (2004) criticises the surrogate production function without postulating that all techniques produce the same basket of goods as net output. Instead, he compares techniques that produce different outputs of the same value, minimising the cost of capital. The physical composition may therefore change along an isoquant to any extent, provided only that the value of output remains the same and capital cost is minimised. He finds on this basis by means of simulations that the conditions for the construction of a production function are unlikely to be fulfilled. This is no surprise at all and will not trouble neoclassical economists, since he has changed the meaning of substitution. Substitution e.g. of capital for labour along an isoquant is a technical replacement of labour by means of instruments in order to obtain the same output. If only the value is kept constant, the notion of substitution loses its meaning. Under these circumstances, the surprise is not that only forty per cent of the cases considered exhibit neoclassical properties; it is rather that as many as 40 % of the cases are neoclassical. The critic is not free to change the conditions of the thought experiment – certainly not without discussing changed assumptions. It is quite clear that Samuelson, von Weizsäcker, Sraffa and others understood substitution to mean switches of means of production and labour to produce the same *physical* output. Sraffa e.g. considers two methods to produce the same commodity when analysing switches throughout chapter XII of his book (Sraffa 1960). Zambelli, by contrast, to put it now more formally, compares two techniques $(\mathbf{A}^{(i)}, \mathbf{l}^{(i)})$; $i = 1, 2$; at a switchpoint $r^*$ such that $\mathbf{p}^{(1)}(r^*) = \mathbf{p}^{(2)}(r^*)$ in the given numéraire, by assuming activity levels $\mathbf{q}_{(1)}, \mathbf{q}_{(2)}$ that minimise capital values, given the value of output $Y = \mathbf{q}_{(1)}(\mathbf{I} - \mathbf{A}^{(1)})\mathbf{p}^{(1)} = \mathbf{q}_{(2)}(\mathbf{I} - \mathbf{A}^{(2)})\mathbf{p}^{(2)}$. Therefore $\mathbf{q}_{(i)}$ is an element of

$$\{\bar{\mathbf{q}}_{(i)} \in Q_i \mid \bar{\mathbf{q}}_{(i)}\mathbf{A}\mathbf{p}^{(i)} \leqq \bar{\bar{\mathbf{q}}}_{(i)}\mathbf{A}\mathbf{p}^{(i)} \textit{ for all } \bar{\bar{\mathbf{q}}}_{(i)} \in Q_i\},$$

where $Q_i = \{\mathbf{q} \geq 0 \mid \mathbf{q}(\mathbf{I} - \mathbf{A}^{(i)})\mathbf{p}^{(i)} = Y\}$ (Zambelli 2004, p. 105). The switchpoint is thus seen as a point on an iso-capital-cost line, not on an isoquant. Along what is however called 'isoquant', consumers in the stationary state do not get the same quantities for consumption but adjust the proportion, in which they consume, so as to minimise something; in Zambelli they minimise not the total cost of production, but the cost of capital. What a strange idea – quite original, but wrong! Consumers can force entrepreneurs to produce cheaply through competition. Entrepreneurs cannot choose the composition of output, which essentially depends on demand, but they can try to produce this output in an efficient manner by minimising total cost. The mistake thus is double: the 'isoquant' is not a line of constant quantity, but constant cost, and cost is erroneously defined as capital cost.

With these assumptions, not only the existence of an old neoclassical equilibrium with non-degenerate solutions – $w$ and $r$ are always positive and there is full employment of both factors – but also that of the production function in per capita terms is implied, for we have, after approximating the envelope by a smooth function, on the one hand for output per head,

$$y(r) = \hat{w}(r) + rk = \hat{w} - r\hat{w}', \tag{19}$$

on the other

$$k(r) = -\hat{w}'(r). \tag{20}$$

(19) and (20) are a parametric representation of the per capita production function $y = f(r)$; the marginal productivity condition is fulfilled because (19) and (20) give

$$\frac{dy}{dk} = \frac{dy/dr}{dk/dr} = \frac{\hat{w}' - \hat{w}' - r\hat{w}''}{-\hat{w}''} = r. \tag{21}$$

The analysis of the old neoclassical equilibrium thus has led us back to the construction of the surrogate production function and to the conditions of its existence. We repeat: the random matrices again come in. The wage curve will be approximately linear under the conditions derived in the first section of this paper. On the one hand, the subdominant eigenvalues must tend to vanish, and this will be the case in the limit for random matrices. On the other hand, we need the covariance condition for the numéraire so as to obtain equations (11) and (12). The condition must hold for all techniques whose wage curves appear on the envelope.

A sacrifice had to be made in the transition from the old neoclassical equilibrium theory to the construction of the surrogate production function. The demand for consumption goods $\mathbf{z}(r)$ changes with distribution, and with it the composition of output $\mathbf{y}_\sigma(r)$, but a clear analysis of the technology is facilitated if $\mathbf{y}_\sigma(r)$ is rigidly given by the numéraire vector $\mathbf{d}$. As we saw, the restriction is not strong in the analysis of an old neoclassical equilibrium, since such an equilibrium can be analysed locally. If an equilibrium has been determined, its stability properties depend on the local properties of the wage curve of the cost minimising technique. Its composition of output can be used as the numéraire and the neighbouring techniques will have a composition of output which is almost the same for reasons of continuity, so that it does not matter much, if this numéraire is kept constant in a full neighbourhood for stability analysis. Formally, output per head is in equilibrium for a technique with a random matrix of type $\mathbf{ce}$ in the limit equal to (compare equations 12 and 14)

$$y=\frac{\mathbf{z}_\sigma\bar{\mathbf{p}}_\sigma}{\mathbf{y}_\sigma\mathbf{l}^\sigma}=\frac{1}{\mathbf{y}_\sigma\mathbf{l}^\sigma}\frac{\mathbf{z}_\sigma\mathbf{x}_\sigma^1}{\mathbf{e}\mathbf{x}_\sigma^1}+(1-\rho\mu_1^\sigma)\frac{\mathbf{z}_\sigma\mathbf{v}_\sigma}{\mathbf{e}\mathbf{x}_\sigma^1}. \tag{22}$$

Hence $\mathbf{z}_\sigma\bar{\mathbf{p}}_\sigma$ will not only change slowly because of continuity. It will actually be constant and independent of $r$, if $\operatorname{cov}(\mathbf{z}_\sigma,\mathbf{v}_\sigma)=0$, since $n\bar{v}_\sigma=0$ (10). If we add the continuity assumptions, we can conclude that the old neoclassical equilibrium and the production function are closely linked concepts.

## 6. Capital in the wage curve diagrams and the production function

We have provided an existence proof for the old neoclassical equilibrium, we have analysed its macro stability, focusing on normal solutions, and this leads us back to the paradoxes of capital, since these are responsible for an essential kind of disequilibrium, which occurs if the capital-labour ratio increases with the rate of profit. Hence we return to the paradoxes and the likelihood of their occurrence in this chapter. In particular, we shall indicate a new way to assess the likelihood of Wicksell effects and compare it to that of reverse capital deepening. The presentation now focuses on the production function.

The odd peculiarity of the old neoclassical equilibrium consists in the assumption of an arbitrary amount of capital $K$ in terms of numéraire, given prior to the determination of prices and of its purchasing power. Since Lindahl (Garegnani 1976), neoclassical economists have abandoned this assumption in the context of general equilibrium, where a vector of endowments of labour, land and capital goods is available in the beginning of a finite or infinite intertemporal series of equilibria in successive periods. The turnpike results suggest that such an economy will converge towards a stationary state, if preferences and the supply of land and labour remain stationary. The crucial conditions (15.5) and (15.12) will then tend to be fulfilled only in the limit. The analysis will therefore eventually also be relevant for intertemporal equilibrium, but this line of thought will not be pursued here. For the importance of the paradoxes of capital is assessed most easily, if the quantity of capital is given in value terms and not as a vector of endowments of physical capital goods as in Schefold (1997) and (2008).

The assumption of a given quantity of capital in value terms facilitates the determination of distribution according to neoclassical principles; it follows that the level of activity, at which the system operates, also is determined on the supply side, not by demand; the same, incidentally, is true for the intertemporal variant of the theory (Schefold 1997, chapter 18).

This is illustrated most clearly, if there is only one technique and if the composition (but not the level) of output is given. Suppose that the wage curve exhibits a neoclassical Wicksell effect as in diagram 1. There is full employment

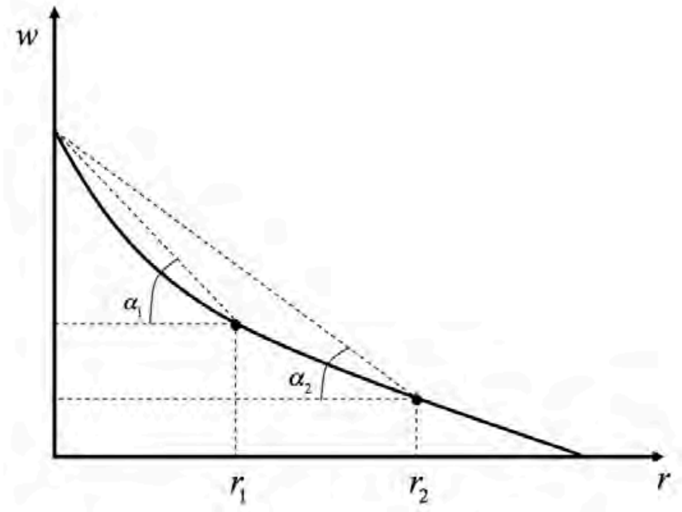


*Diagram 1*: Wage curve for one technique with neoclassical Wicksell effect. Full employment of labour force. $L_1$ at $r_1$, of labour force $L_2 > L_1$ at $r_2$. Capital intensities $k_1, k_2$ given by $tg\alpha_i = k_i$.

with labour force $L_1$ at $r_1$, and $k_1 = K / L_1$, $K$ being the given amount of capital. An immigration takes place, $L_1$ rises to $L_2$, unemployment arises, the wage rate falls. The shape of the wage curve implies that the intensity of capital falls as $w$ falls and $r$ rises, until some $r_2$ is reached, where $k_2 = K / L_2$. At this level, the intensity of capital has fallen to such an extent that the given amount of capital suffices to employ $L_2$. The argument runs from the amount of capital via distribution to the level of activity. The amount of capital is in value terms. The technique has not changed, the composition of output not either. It is possible to employ more people, just because relative prices change in such a way with distribution that the physically same capital costs less, hence more of it can be produced, given $K$, and activity levels rise in the same proportion. A displacement of an old neoclassical equilibrium because of an increased labour force would look the same, if the composition of demand happened to stay constant.

A Keynesian would see a different causal mechanism at work, while he observed the same facts: An immigration occurs and depresses wages. Fortunately, for no apparent reason, effective demand rises. Full employment is reached, only by coincidence the same amount of capital is employed.

Now consider the opposite case of an anti-neoclassical Wicksell effect as in diagram 2.

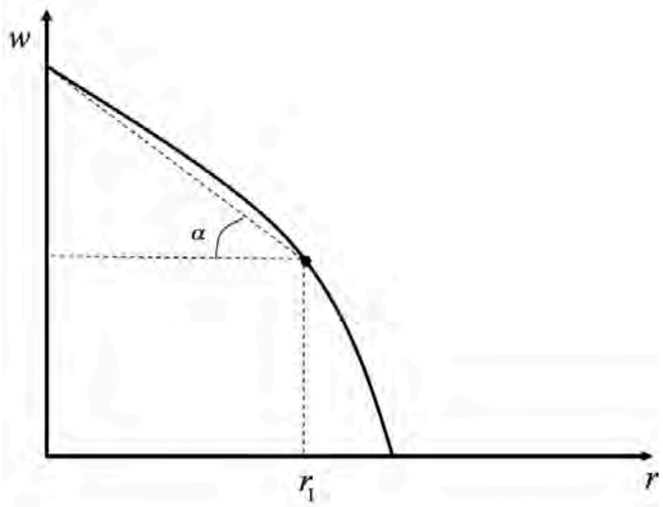


*Diagram 2*: A wage curve exhibiting an anti-neoclassical Wicksell effect.

Let the economy be in conditions of full employment at $r_1$, with intensity of capital $k_1$. If the employment is disturbed by a possibly small immigration, wages will begin to fall. Here, the intensity of capital rises because of the shift of relative prices with distribution. Given $K$, the demand for labour $L = K / k$ will fall and depress wages even more; they fall below subsistence and tend to zero; the system is unstable.

A Keynesian, who believes in sticky money wages, need not accept this consequence. If prices are also sticky, real wages do not change at $r_1$, and if effective demand can be made to rise, unemployment can be absorbed without a change of distribution, but more investment is necessary, so that the value of capital rises in step with the labour force.

It may be remarked here that not all Keynesian propositions remain unaffected by the critique of capital, as I have shown elsewhere (Schefold 1979). For instance, the Kaldorian theory of steady growth assumes a constant capital-output ratio. This means that the wage curve must be a straight line which turns around the maximum rate of profit (see diagram 3) with neutral technical progress.

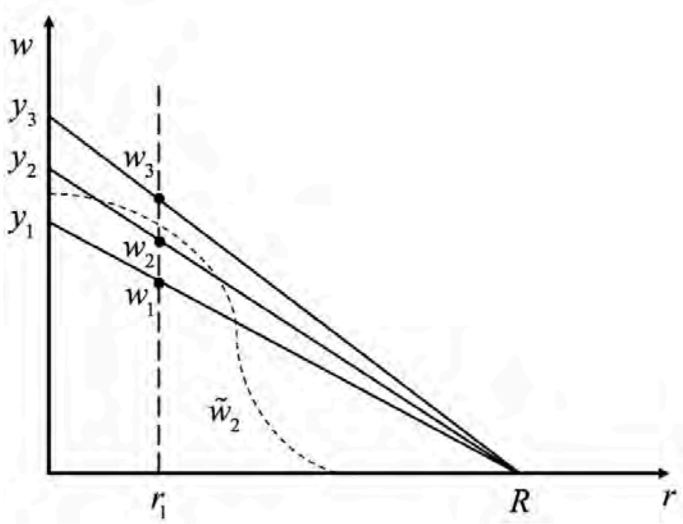


*Diagram 3*: Steady growth according to the stylised facts. Wage curves $w_1, w_2, w_3$ exhibit a regular growth of productivity, with wages at $r_1$ rising in step with output per head. If the technique in the second period is not represented by $w_2$ but by $\tilde{w}_2$, which is more profitable than $w$ at $r_1$, output per head falls, because of reverse capital deepening.

But reverse capital deepening could upset the process, as the diagram shows. Cambridge critics of neoclassical theory who assert that capital reversals occur sufficiently often to be a problem for their opponents must explain why they exclude such phenomena from their own theory.

We get out of the dilemma by recognising that capital reversals are rare. I have argued this in earlier papers, which have been mentioned, on the basis of empirical analyses using input-output tables and – in order to explain the empirical evidence theoretically – by means of the theory of random matrices, as in section 2 of this paper. Here I want to provide a reasoning as to why both neoclassical and anti-neoclassical Wicksell effects occur more frequently than reverse capital deepening, and why reswitching is so rare as to be negligible.

A measure for the likelihood of reswitching was given in Schefold (1976a) by considering the set $M_1$ of conceivable methods of production $\{(\mathbf{a}_0, l_0) \geq 0\}$, which are an alternative to the method employed in the first industry $(\mathbf{a}_1, l_1)$ of a given system $(\mathbf{A}, \mathbf{l})$ by having a switchpoint – that is by having the same price for the first commodity – at a given rate of profit $r_1$. Using

$$M(r) = \{(\mathbf{a}_0, l_0) \geq 0 \mid (1+r)\mathbf{a}_0\hat{\mathbf{p}}(r) + l_0 = (1+r)\mathbf{a}_1\hat{\bar{\mathbf{p}}}(r) + l_1, 0 \leq r < R\}, \hat{\mathbf{p}} = p / w.$$

we find that $M_1 = M(r_1)$; this is a simplex of dimension $n$ in $I\!R^{n+1}$. Consider $M(r_1) \cap M(r_2), r_1 \neq r_2$. This is an intersection of two simplices of dimension $n$ in $I\!R^{n+1}$. The simplices are different, if $\hat{\mathbf{p}}(r_1)$ is not proportional to $\hat{\mathbf{p}}(r_2)$, hence essentially, if prices are not proportional to labour values. For more details and a diagrammatic representation (fig. 1) see Schefold (1976a).

$M(r_1) \cap M(r_2)$ is of dimension $n-1$. It is trivial that $M(r_1) \cap M(r_2)$ contains $(\mathbf{a}_1, l_1)$ for all $r_2$. Let $\mu(M)$ be the $n$-dimensional Euclidean measure of set $M$. Obviously $\mu(M_1) > 0$, but $\mu\left(M(r_1) \cap M(r_2)\right) = 0$. This means that a given alternative method $(\mathbf{a}_0, l_0)$ at $r_1$ will only by a fluke be an alternative also at $r_2$. To get reswitching at two pre-assigned rates of profit is possible only by a fluke. But $M(r_1) \cap M(r_2)$ turns around $(\mathbf{a}_1, l_1)$ as $r_2$ varies and thus covers open $n$-dimensional neighbourhoods. $(\mathbf{a}_1, l_1)$ is semi-positive, but not necessarily positive. The case where $(\mathbf{a}_1, l_1)$ is not strictly positive has been drawn in fig. 1 in Schefold (1976a). $M(r_1) \cap M(r_2)$ then covers a triangle, if $n = 2$. Generally, the likelihood of getting a reswitch somewhere in $\{0 \leq r_2 < R\}$, $r_2 \neq r_1$, if we let $r_2$ vary, given $r_1$, is not a fluke, since

$$M^* = \bigcup_{\substack{0 \leq r_2 < R \\ r_2 \neq r_1}} \left[M(r_1) \cap M(r_2)\right]$$

is $n$-dimensional and the larger, the larger is the movement of relative prices. The ratio

$$\pi = \frac{\mu\left(M^*\right)}{\mu\left(M(r_1)\right)}$$

can be interpreted as the likelihood of reswitching. Clearly, $\pi > 0$, if relative prices are not constant, but the change of relative prices in $(0, R)$ is limited; so $M(r_1) \cap M(r_2)$ covers only a small part of $M(r_1)$ as $r_2$ varies. If the change of relative prices is bounded, it turns out that, the larger the dimension of the system, the smaller is the volume of the set of potential methods which give rise to reswitching relative to the volume of the set of all the potential methods which give rise to one switch. Hence, $\pi$ must be much smaller than one for any given system with many sectors. One could also show that $\pi$ diminishes, as the system approximates random properties, but we already know that the wage curves will then tend to be linear, so that $\pi$ then tends to zero. For more details see the Appendix.

Reswitching has been observed empirically only once in the literature (Han and Schefold 2006). The reason is that the lower of the two switchpoints will usually be found below one or several wage curves; one then speaks of reverse capital deepening (diagram 4). But reverse capital deepening also is rare, and for the same reasons as reswitching. Note that

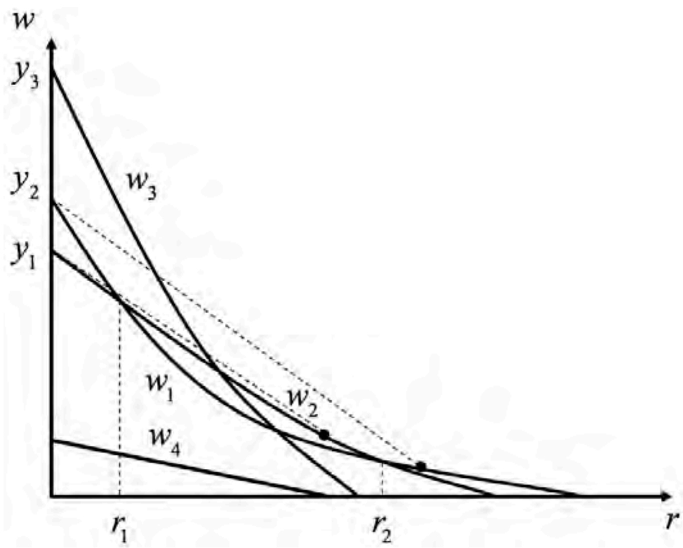


*Diagram 4*: Reverse capital deepening. The techniques represented by $w_1$ and $w_2$ differ only in the method of production in one industry. Hence it would be a matter of reswitching with two switches on the envelope at $r_1$ and $r_2$, but the switch at $r_1$ is dominated by $w_3$. There is a capital reversal at $r_2$ in that $k_1(r_2-\varepsilon)<k_2(r_1+\varepsilon)$ for $\varepsilon\to 0, \varepsilon>0, k_i(r)=(1/r)\left(y_i-w(r)\right)$. Finally, $w_4$ is an inefficient, but very labour-intensive technique, discussed in section 7 below.

the rates of profit at which reswitching or reverse capital deepening takes place are independent of the numéraire. Note further that this capital reversal has nothing to do with a specific form of the Wicksell effect, in that one, two or all three of these curves could also exhibit anti-neoclassical Wicksell effects in this diagram.

We now attempt (for the first time, as far as I know) an attempt to provide an estimate for the likelihood of anti-neoclassical Wicksell effects. We keep the assumption that the net output $\mathbf{d}$ is taken as the numéraire. Whereas reswitching and reverse capital deepening are invariant, if $\mathbf{d}$ changes, not only the direction but also the magnitude of Wicksell effects depends on $\mathbf{d}$. This may be illustrated as follows, using the standard commodity $\mathbf{s}$ of a given technique ($\mathbf{A},\mathbf{l}$), with $\mathbf{s}=\mathbf{q}(\mathbf{I}-\mathbf{A})$, with $(1+R)\mathbf{qA}=\mathbf{q}>0$, $\mathbf{ql}=1$ and, in consequence, the linear wage curve $w=1-(r/R)$. It follows that the intensity of capital equals $1/R$. This means that there is no Wicksell effect, because the standard commodity is also the net output. But by varying net output and the numéraire, the same technique will give rise to *both* Wicksell effects. Relative prices never are constant in regular systems. Hence, starting from the situation in which the standard commodity is numéraire, we can always locally produce either a neoclassical or an anti-neoclassical Wicksell effect by slightly changing the quantities contained in the numéraire in one direction or the other. Wicksell effects result from a choice made by the observer regarding the numéraire to be used.

For easier comparison, we limit the analysis to wage curves with $w(0)=1$. The maximum wage rate and the maximum rate of profit of all these wage curves resulting from a variation of $\mathbf{d}$ then remain the same. Now it is clear that such a wage curve must at least for some range of the rate of profit exhibit a neoclassical Wicksell effect, if $w(r)<1-(r/R)$ and an anti-neoclassical Wicksell effect, if $w>1-(r/R)$; this criterion will be unambiguous, if there is no inflection point (neoclassical: $w''(r)>0$; $0\le r\le R$; and anti-neoclassical: $w''(r)<0$; $0\le r\le R$); see diagram 5.

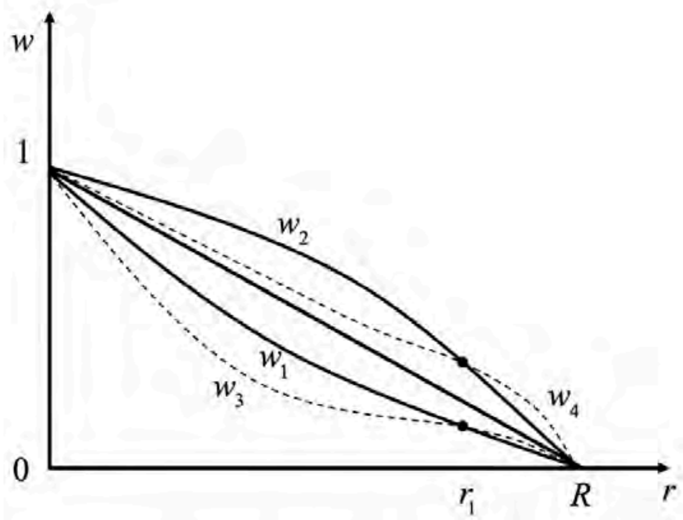


*Diagram 5*: The same technique gives rise to different Wicksell effects, represented relative to the standard wage curve: $w_1$ neoclassical throughout, $w_3$ with inflection point at $r_1$ in part neoclassical ($r$ low); $w_2$ anti-neoclassical throughout, $w_4$ in part anti-neoclassical ($r$ high).

Hence, choosing some $r_1$; $0<r_1<R$; we can divide the set $D$ of all numéraires/net outputs for technique ($\mathbf{A},\mathbf{l}$), with $w(0)=1$, into three sets (assuming that $\hat{\mathbf{p}}(0)$ and $\hat{\mathbf{p}}(r_1)$ are not proportional):

$$D_s=\{\mathbf{d}\in D/w=1-r_1/R\},$$
$$D_n=\{\mathbf{d}\in D/w<l-r_1/R\},$$
$$D_a=\{\mathbf{d}\in D/w>l-r_1/R\}.$$

The numéraires/net outputs yield wage curves that are at least in part associated with neoclassical Wicksell effects in $D_n$, with anti-neoclassical Wicksell effects in $D_a$ and indeterminate are those in $D_s$. We rewrite the definitions, using the hyperplane $D_0 = \{\mathbf{d} \mid \mathbf{d}\hat{\mathbf{p}}(0) = 1\}$ and remembering $w(r) = 1/\mathbf{d}\hat{\mathbf{p}}(r)$.

$$
\begin{aligned}
D_s &= \{\mathbf{d} \geq 0 \mid \mathbf{d}\hat{\mathbf{p}}(r_1) = R/(R-r_1)\} \cap D_0,\\
D_n &= \{\mathbf{d} \geq 0 \mid \mathbf{d}\hat{\mathbf{p}}(r_1) < R/(R-r_1)\} \cap D_0,\\
D_a &= \{\mathbf{d} \geq 0 \mid \mathbf{d}\hat{\mathbf{p}}(r_1) > R/(R-r_1)\} \cap D_0.
\end{aligned}
$$

The numéraires/net outputs are therefore all on the simplex of the semi-positive vectors on the hyperplane of dimension $n-1$, $D_0$, which is orthogonal to $\hat{\mathbf{p}}(0)$, $D_s$ are the semi-positive vectors on the hyperplane of dimension $n-2$ on $D_0$, which results from the intersection of $D_0$ and the hyper plane of vectors orthogonal to $\hat{\mathbf{p}}(r_1)$. $D_s$ partitions $D_0$ into $D_n$ and $D_a$. The intersections are not empty, since $\mathbf{s} \in D_s$ and $\mathbf{s} > 0$, hence $\mu(D_n) > 0, \mu(D_a) > 0$: *one and the same technique* $(\mathbf{A}, \mathbf{l})$ *always yields both neoclassical and anti-neoclassical stretches of wage curves, depending on the choice of the numéraire/net output* (see diagram 6), provided only that relative prices at $r = 0$ and $r = r_1$ are different, i.e. provided that the labour theory of value does not hold.

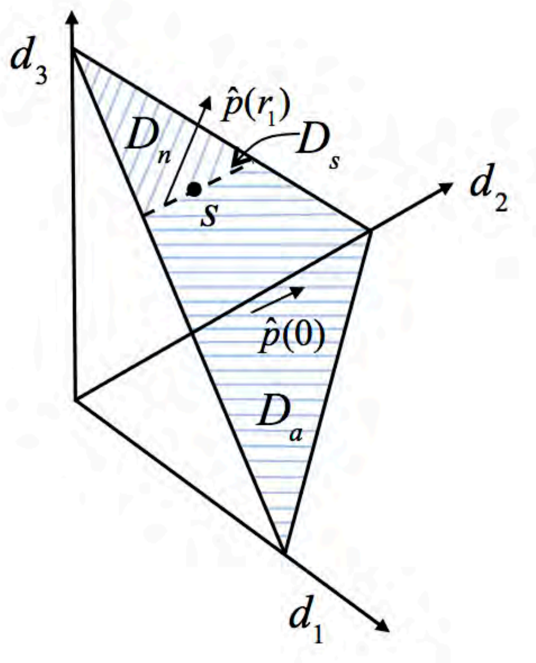


*Diagram 6*: The numéraire vectors that give rise to neoclassical ($D_n$) and to anti-neoclassical Wicksell effects ($D_a$).

If prices are proportional to values, $D_s$ coincides with $D_0$ and Wicksell effects are not observed. If prices are close to values, the Wicksell effects are small, and, as always, of both kinds depending on $\mathbf{d}$. If $\hat{\mathbf{p}}(0)$ is not far from being proportional to $\mathbf{e}=(1,...,1)$, $\hat{\mathbf{p}}(0)$ is close to the „middle“ and $\hat{\mathbf{p}}(r_1)$ as well. In consequence, $\mu(D_n)$ and $\mu(D_a)$ will not be very different. The likelihood of the anti-neoclassical case could be expressed by

$$\varphi = \frac{\mu(D_a)}{\mu(D)}.$$

Clearly $0<\varphi<1$, since $\mathbf{s}\in D_s$, $\mathbf{s}>0$, unless the labour theory of value holds, and $\varphi$ could be close to 1/2 in most cases; hence $\varphi$ tends to be much larger than $\pi$, the likelihood of a capital reversal due to reswitching or reverse capital deepening.[10]

The Wicksell effects considered so far are measured at $r_1$ relative to $w(0)=1$ and $w(R)=R$. Their absence means that the value of capital per head at $r_1$ is the same as if the wage curve was linear between $w(0)$ and $w(R)$, hence $k(r_1)=1/R$. Wicksell effects are absent for regular systems at *all* rates of profit between zero and the maximum, if and only if the numéraire is proportional to the standard commodity and, with our normalisations, equal to $\mathbf{s}$, shown in diagram 6 on point $S$ on $D_s$. As is well known, we have $\mathbf{s}>0$, hence we see again that $S$ is an inner point of $D_0$, and the actual numéraire may just as well be in $D_a$ or $D_n$. To say that Wicksell effects are positive or negative with probability one is to repeat the familiar insight that Sraffa systems are regular in the sense of Schefold (1971) with probability one. Are the effects likely to be large on the envelope? That depends on the rapidity with which prices change from one switchpoint to the next; they will change little, if the system approximates random properties. This does not lead to a new insight: we already know that random systems tend to have linear wage curves according to equation (11), so that Wicksell effects are excluded. Moreover, it is important to note that Wicksell effects are small in the neighbourhood of $r=0$ (equation 12) in random systems. By contrast, the non-

[10] This assertion is based on geometric evidence, comparing diagram 6 with fig. 1 in Schefold (1976a), and on taking into account the effect of increasing $n$. Note that Sraffa systems are regular (have changing relative prices with probability one), hence Wicksell effects exist with probability one (they are excluded (are equal to zero) only if the numéraire is on $D_s$ (diagram 6). Only the magnitude of the Wicksell effect will be small, if the change of relative prices is small. Some Wicksell effect, small or large, neoclassical or anti-neoclassical will almost surely occur. By contrast, and as we have seen, the likelihood of reswitching itself, not only the magnitude of the change in the value of capital, will be small, if the change of relative prices is small, as follows from the argument in Schefold (1976a). The reader is advised also to consider the geometry of fig. 1 in Schefold (1976a), if $n=3$, and of diagram 6 here, if $n=4$.

dominant eigenvalues play a significant role for Wicksell effects near $r=0$, if these eigenvalues are large enough (equation 7). Nevertheless, truly large Wicksell effects are likely to be encountered only at large rates of profit.

We turn to the envelopes of the wage curves derived from a large spectrum of techniques such as that resulting from combining the methods of $h$ countries with $n$ industries each. They all must produce the same net output – otherwise the purpose of the comparison, the determination of the best technique, given $r$, to produce the same output, is missed. Taking the output as the numéraire allows to apply the above analysis. That both kinds of Wicksell effects are about equally likely does not mean that neoclassical and anti-neoclassical Wicksell effects will sort of alternate. For technical change along the envelope is piecemeal, i.e. only one method at each switchpoint is changed at a time. Such a method switch affects only $n+1$ coefficients of the $n(n+1)$ coefficients of $(\mathbf{A},\mathbf{l})$ and cannot drastically alter the properties of the individual wage curves and, in particular, their second derivative, if $n$ is large enough – two and three sector models are misleading here. Hence, we must expect that one group of wage curves exhibiting one kind of Wicksell effect will be followed by another group exhibiting the other kind, as one moves along the envelope, with $w''$ falling or rising stepwise and changing sign not often.

It then becomes important to clarify what one means by „large“ or „small“ Wicksell effects. Visual inspection of a diagram will not suffice, since the same $w(r)$ may appear strongly or weakly curved depending on the scale chosen for measuring $r$ or $w$, given a scale for measuring $w$ or $r$. The obvious choice is to speak of a value elasticity $\eta$ of the intensity of capital $K/L$ with respect to distribution expressed by $r$ or $w$. The ordinary elasticity of a demand curve is an analogue. If $r$ is replaced by $\lambda r,\ \lambda>0,$ a Wicksell effect seems to get reduced or magnified in the diagram, but $\lambda$ cancels out in $dr/r$. The value elasticity $\eta$ thus is

$$\eta=\frac{dk/k}{dr/r}=\frac{dk}{dr}\frac{r}{k}.$$

Since $k(r)$ depends in an essential way on the numéraire, which is the net output, we do not need an analogue of the more complicated elasticity of substitution, which would depend on $w/r$.

The hyperbolic wage curves

$$w(r)=\frac{R-r}{R+\alpha r}$$

may serve as an example with $w(0)=1, w(R)=0$ for all $\alpha > -1$. If $\alpha = 0$, we get the straight standard wage curve. The neoclassical Wicksell effect results, if $\alpha > 0$, the anti-neoclassical for $-1 < \alpha < 0$. One finds $k = (1+\alpha)/(R+\alpha r)$ and

$$\eta = -\frac{\alpha r}{R+\alpha r}.$$

We have $\eta = 0$ and $k = 1/R$ for $\alpha = 0$. The intensity of capital rises by $\eta$ percent, if $0 > \alpha > -1$ and $r$ increases by one percent.

The following transformation using the definition of $k$ yields a familiar geometric interpretation of $\eta$:

$$\begin{aligned}\eta &= \frac{r}{k}\frac{dk}{dr} = r\frac{r}{w(0)-w(r)}\frac{w'(r)r-\big(w(0)-w(r)\big)}{r^2} \\ &= \frac{w'r}{w(0)-w(r)} - 1 = \frac{w'}{k} - 1 = \frac{tg\beta}{tg\gamma} - 1.\end{aligned}$$

where $\beta$ is the angle indicating the absolute value of the shape of $w(r)$ at $r$ and $\gamma$ is the slope of the straight line connecting $w(0)$ and $w(r)$ as in diagram 7, obviously $\beta > \gamma$

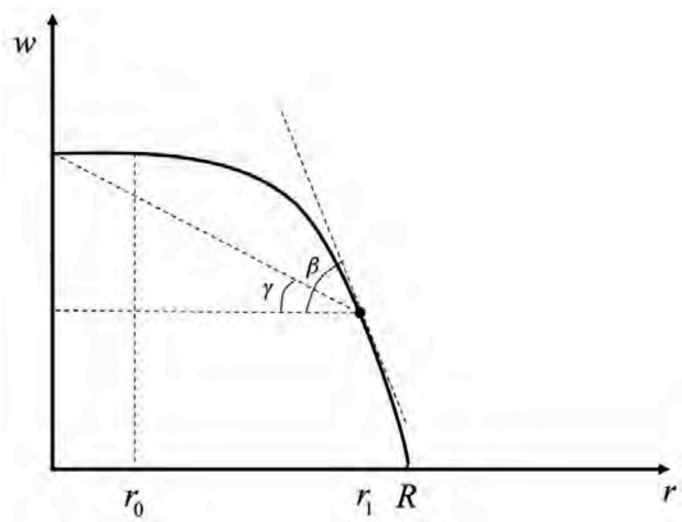


*Diagram 7*: The value elasticity of the intensity of capital $\eta = (dk/dr)(r/k)$ equals $(tg\beta/tg\gamma)-1$. The elasticity is small at low rates of profits such as $r_0$.

for anti-neoclassical, $\beta < \gamma$ for neoclassical Wicksell effects. If the angles are not large, $\beta / \gamma - 1$ will do as an approximation for $\eta$. There is no Wicksell effect and $k$ is locally constant where $\beta = \gamma$ and $\eta = 0$. The value elasticity tends to zero as the rate of profit tends to zero. If the scale for measuring the wage is changed by applying a factor of proportionality, given a scale for measuring the rate of profit, capital-intensities and differences between them will also be increased or diminished proportionally, but not so the elasticity, which stays invariant.

The application of the theory of random matrices suggests that inflection points of wage curves will be rare so that the anti-neoclassical Wicksell effect must look like the wage curve in diagram 7. A strong anti-neoclassical Wicksell effect in the sense of $\eta$ being large can be expected only at high rates of profit.

We now can pull the strings together and describe the likely characteristics of the envelope of the wage curves derived from a large but finite spectrum of techniques such as the one introduced in section 3 which consists of the different methods employed for the production of $n$ goods in $h$ countries so that the number of combinations is $s = h^n$; e.g. $h = 10, n = 100$, so that $s = 10^{100}$. The likely properties of the envelope are, assuming a common net output taken as a numéraire:

1. No wage curve is exactly straight, but most are close to linearity, especially at low rates of profit, since the input-output coefficients are approximately random and the covariance conditions approximately fulfilled.
2. Inflection points are not frequent.
3. Only a tiny number of the $s$ wage curves appear on the envelope; all others are below. The expected number of wage curves on the envelope is smaller than $\ln s = n \ln h$, where $\ln$ is the natural logarithm (Schefold 2013b).
4. If the numéraire is a single commodity, the deviation of a wage curve from linearity can be dramatic, but this is irrelevant for the comparison of the economies of competing nations with positive vectors for the common net output serving as a numéraire.
5. Reswitching will hardly ever be observed and reverse capital deepening will be rare.
6. Since the wage curves are not exactly straight, there will always be Wicksell effects, both anti-neoclassical and neoclassical. The same technique can result in both kinds of Wicksell effects, depending on the net output taken as numéraire.
7. Curves exhibiting anti-neoclassical and curves exhibiting neoclassical Wicksell effects will appear in groups, as one moves along the envelope. The wage curves in the transition between these groups are close to linearity.
8. The value elasticity of the intensity to capital with respect to changes in distribution will tend to be higher at high, and lower at low rates of profit.
9. The maximum of the maximum rates of profit $R_\tau$ will be quite high, since we have here only taken circulating capital into account.

10. Since the envelope extends far, with $R_\tau$ above 100 %, only few wage curves are present and deviations from linearity must be small in the relevant range of $r$ of perhaps 5 % to 15 %.

## 7. Conclusions: The improbability of reswitching, the certainty of Wicksell effects and the poverty of production functions

The results presented in the last section, obtained by combining new theory with plausible extrapolations, seem to me to be confirmed by the empirical results by Mariolis and Tsoulfidis for individual wage curves of many countries[11] and by Zambelli (2014) for the only major cross-country comparison I know. He obtains an envelope of wage curves for 30 countries on the basis of input-output tables for 35 industries. A visual inspection of his remarkable wage curve diagram seems to me to confirm the 10 points made above. The wage curves are almost straight in the relevant range, the Wicksell effects come in groups, nearly linear curves between them, and his envelope extends over more than 250 % (his $R_\tau$ is about 250 %), hence, with 63 switchpoints appearing on the envelope, the average distance between switchpoints is large, near 4 %. There is by no means an avalanche of switchpoints, as one runs down the envelope. According to the formula for the upper limit of the expected number of wage curves on the envelope already referred to, $\ln s = n \ln h$, proved in Schefold (2013b), the number of switchpoints on the envelope must be smaller than $31 \cdot 3.5 = 108.5$, which is in agreement with Zambelli's result and a small number, considering that the spectrum consists of about $6.18 \cdot 10^{45}$ techniques. But he uses other measures than those employed here e.g. for assessing the Wicksell effects, and so no agreement could be reached in our personal debate on the interpretation of his results. I understand that he is still working on it.

Meanwhile a profound modification of the old Cambridge critique of neoclassical theory takes place. The argument, neoclassical theory is entirely wrong because of reswitching etc., cannot be sustained except at the level of the most abstract theorising, while new arguments are coming up. The core of the neoclassical propositions is: if there is unemployment in a closed economy, a fall of real wages will make *known* methods of production profitable which will use more labour; the conversion of existing capital goods into others with a lower intensity of capital will make it possible to absorb the unemployed without a (significant) volume of net investment. The production function postulates that there is an infinity of substitution possibilities in any small intervall, a surrogate production function postulates that the possibilities are many. (In the continuum, the switchpoints disappear, there are only techniques.) On Zambelli's

[11] See e.g. Maridis and Tsoulfidis (2014) and the references therein to papers by the same and other authors, who arrived at conclusions similar to the ones presented here, but by a different road.

envelope, the wage has on average to fall to such an extent that the rate of profit rises by 4% for another efficient technique to be reached by one single switch, and the change of technique will take place in only one industry. It is true that the switch in itself will be neoclassical, the intensity of capital will fall, but perhaps only by little, and the new technique – and also the old – may exhibit an – in the relevant range probably small – anti-neoclassical Wicksell effect.

Nevertheless, it cannot be denied that such labour-intensive methods exist. If wages fall low enough, the spade could be used instead of the plough and the tractor, but it would be an inefficient technique and the corresponding wage curve far below the envelope: not only the maximum wage at $r = 0$ would be very low, but also the maximum rate of profit, which is a measure of the maximum rate of growth. No developed country can return to the techniques of 1960 because it was nearer full employment then than today. Our analysis suggests that the neoclassical mechanism to ensure full employment is no so much threatened by capital reversals, creating instabilities, as by a lack of suitable *efficient* techniques, that is, by a lack of wage curves *on the envelope*, which stand for techniques allowing to increase employment significantly, using existing capital resources which are transformed somehow in accordance with Clark's vision. For we have seen that, at the relevant low rates of profit, wage curves appear on the envelope with properties that do not deviate much from the neoclassical ideal, but they are few. But below the envelope, there is an unknown multitude of techniques, and doubtless many of them are more labour-intensive. They are many, if we look at the combinations of methods in actual use in the countries under consideration. There are many more, if we include techniques that had once been in use in earlier times. We only have to look back in economic history to remember them. Or we might include labour-intensive methods used in less developed countries. Wage curve $w_4$ in diagram 4 in section 6 illustrated such an inefficient, but very labour-intensive technique.

"Set fire to the new factories! Break the machines!" was the battle cry of the Luddites. Are the neoclassicals Luddites in disguise? To give in to the tendency to use inefficient techniques to absorb unemployment is not more – rather less – absurd than the proposal to produce something not really useful in order to create employment – there are traces of the pursuit of both strategies in many countries. Of course, the neoclassicals do not want to be Luddites; they hope for efficient techniques. But these may not lead to significant increases of employment with given resources, and our possibilities to influence the directions of technical change remain quite limited. The Keynesians want meaningful investment projects, but they are not easy to identify and to finance. It is a challenge to find ways to expand effective demand for useful purposes.

We return to the still broader historical perspective. The last section has shown drastically, the section on the old neoclassical equilibrium more subtly, that $K$ plays the decisive role in the determination of the level of activity in the neoclassical approach. The supply side limits the level of activity, at which the equilibrium settles down. In Keynes, the expansion of output through multiplier processes is limited by effective demand, provided there is unused capacity and a reserve of labour. In Marx, by contrast, the process of accumulation is dynamic. Profits are invested, which are extracted as surplus value in a conflictual confrontation of capital and labour. Profits are the aim, not the satisfaction of the consumers, whereas the firms in neoclassical equilibrium are bound by the readiness of workers to sacrifice leisure time for wages and by savers to provide capital for interest.

Despite the fundamental differences in outlook, these schools of economic thought have the basic elements of the theory of normal prices in common, with shortcomings typical for their time. They all argue as if the paradoxes of capital did not exist, and this is true even for Keynesian approaches. When the paradoxes of capital were brought to the fore in the reswitching debate, it seemed to many that economic theorising would have to start afresh. However, it now appears the paradoxes occur only rarely. Reswitching is improbable. Wicksell effects always exist; if anti-neoclassical, they create an instability, if "neoclassical", they still disturb the idea that mere physical properties could allow to distinguish capital-intensive and labour-intensive techniques. The theoretical economist aiming at empirical relevance will perhaps not be impressed by such Wicksell effects, which have been known to exist since Ricardian times, since prices are somewhat sticky. So it is what we might call the poverty of production functions, the fear that the approximate efficient labour-using technique simply might not exist, that must worry the economist. The new facts that gave rise to this conclusion can be explained by the theory of random matrices. It turns out – and to show it was the purpose of this paper – that this explanation also helps – within limits – to justify theories as diverse as the Marxian, the old neoclassical equilibrium and the modern variant of Clarkian economics: the production function. Some arguments of the critique have lost their importance, others have gained and new ones have been found: It is a new departure.[12]

[12] What we have found here, may appear in a different light, if the assumptions are varied once more. We based the analysis on industries represented at an intermediate level of aggregation. What will come out, if the level of aggregation is much lower so that sparse matrices are encountered? We then come back to the circular matrices mentioned in the first section of this paper, for the extreme form of a sparse indecomposable matrix is circular, with only $n$ elements $a_{1n}, a_{21}, a_{32}, ..., a_{n,\,n-1}$ being positive. It has been shown (Schefold 2010) that such matrices can give rise to strongly curved wage curves, with extreme Wicksell effects; it could be shown that the paradoxes of capital are in this family of matrices almost as easily engendered as in Austrian models without basic commodities.

One might argue that circular matrices would on average also yield linear wage curves like random matrices. But there is a difficulty. If the first $t$ rows and columns of the circular matrix and the last $n-t$ rows and columns are aggregated to form an aggregate two-sector model with input coefficients $\bar{a}_{12}$ and $\bar{a}_{21}$ as the arithmetic means of the first $t$ and the last $n-t$ coefficients of $a_{1n}, a_{21}, a_{32}, ..., a_{n,\,n-1}$ respectively, the aggregate system

## Appendix: More on the likelihood of reswitching and of Wicksell effects

The first numerical illustrations of the paradoxes of capital looked artificial in Pasinetti (1966). Samuelson (1966), eager to correct his mistake, tried to give an intuitive idea why they were possible, but he, like Pasinetti, started from an Austrian approach. The first construction of a numerical example with three switchpoints was similar in character (Schefold 1976b). As mentioned above, I felt that it might be useful to demonstrate that the probability of reswitching was positive and not a fluke and to embed the proof into a full $n$-sector model without Austrian characteristics (Schefold 1976a). Today, after the empirical investigations, which also have been mentioned, we need not demonstrate anymore that reswitching exists or that it is rare, but we have to explain why both findings can be true at the same time. I here want to show that the size of the system (the number of sectors) plays an important, so far neglected, role. The idea is to impose a mathematically simple condition which reflects more complicated

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$$\mathbf{A} = \begin{pmatrix} 0 & \bar{a}_{21} \\ \bar{a}_{21} & 0 \end{pmatrix}, \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

retains the circular structure and will in fact yield a linear wage curve, if on average $\bar{a}_{12} = \bar{a}_{21}$ and if the same holds true for the corresponding aggregate labour inputs. This result follows, although all eigenvalues will be equal in modulus, independently of the numéraire, and it will be obtained for almost all such aggregations, if the coefficients $a_{1n}, a_{21}, a_{32}, ..., a_{n,\,n-1}$ and the labour inputs are identically and independently distributed and $n$, $t$ and $n-t$ are large.

But this procedure to obtain a linear wage curve excludes the choice of technique, if the coefficients of the individual techniques are i.i.d., but with different arithmetic means. For if there are two techniques with input coefficients and labour coefficients, which are i.i.d., the coefficients of any combined technique will here, with two circular systems, not be i.i.d. The arithmetic mean e.g. of the first $t$ coefficients of the first technique will have one arithmetic mean, the $n-t$ following coefficients of the second technique will have another; the combined technique will give rise to a two-sector model with a wage curve, which will not be linear and yet may appear on the envelope, depending on the numéraire, as the reader can verify by means of a short calculation. The random systems, by contrast, yield techniques that are again random – convex combinations of i.i.d. *processes* are i.i.d. –, and that is the fundamental reason why they can be used to construct approximate surrogate production functions. By contrast, combinations of circular i.i.d. *techniques* are circular, but not i.i.d.

Random systems and circular systems are polar cases in the input-output world. If one looks at input-output tables, the former appear as much more plausible and the potential for the paradoxes of capital is reduced, justifying the simplifying approach to normal prices adopted in Marx, in the old neoclassical equilibrium and in the construction of the surrogate production function. However, new results obtained from working with sparse matrices may compel us to accept new conclusions once more. It may not please either side in the debate, but the discussion is still open.

economic conditions: the fact that relative prices change with distribution but only to some extent. The bound means that reswitching is still possible with a finite probability for any given system. It is then shown that this probability shrinks to zero as the size of system increases. The findings of some earlier authors (D'Ippolito 1987, Petri 2011) that reswitching is not quite so unlikely in systems with only a few sectors and the rarity of reswitching results in empirical investigations using large input-output systems therefore do not contradict each other.

We use the notation of section 6. In order to push the analysis of the likelihood of reswitching further, we consider the vertices spanning the simplex $M(r)$ as functions of $r$. They are denoted by $z_i\mathbf{e}_i$; $i=1,...,n+1$; $\mathbf{e}_i$ being unit vectors in $I\!R^{n+1}$.

Let $\bar{\mathbf{p}}, \bar{w}$ be standard prices and wage rate $\bar{w} = 1 - r/R$,

$$\tilde{\mathbf{p}} = \begin{pmatrix} (1+r)\bar{\mathbf{p}} \\ \bar{w} \end{pmatrix}.$$

The vertices of $M(r)$ then fulfil

$$\left[z_i\mathbf{e}_i - (\mathbf{a}_1, l_1)\right]\tilde{\mathbf{p}}(r) = 0; \quad i = 1,...,n+1;$$

so that they can be calculated:

$$z_1 = \frac{1}{1+r},$$
$$z_i = \frac{\bar{p}_1}{(1+r)\bar{p}_i},$$
$$z_{n+1} = \frac{R\bar{p}_1}{R-r}.$$

The $z_i$ are continuous in $0 \le r < R$, the prices also being continuous functions, but $z_{n+1}$ diverges to infinity at $R$. In order to visualise how $M(r)$ shifts with changes of $r$, we locate the vertices of $M(r)$ in $I\!R^{n+1}$ in the two-dimensional coordinate hyperplanes $H_{ij}$ of $I\!R^{n+1}_{+}$. The coordinates of $I\!R^{n+1}$ are $y_1,...,y_{n+1}$; $y_{n+1}$ is the coordinate of the labour input. There are $n(n+1)/2$ such hyperplanes with $i<j$, if one avoids double counting. The line segments $h_{ij}(r)$ connecting $z_i(r)\mathbf{e}_i$ and $z_j(r)\mathbf{e}_j$ in $H_{ij}$ represent the edges of $M(r)$. The movements of $h_{ij}(r)$ show us how the set of potential techniques moves with $r$; intersections $P_{ij}(r_1, r_2)$ of $h_{ij}(r_1)$ and of $h_{ij}(r_2)$ span the convex $(n-1)$-

dimensional set $M(r_1) \cap M(r_2)$ of potential techniques leaving a switch at $r_1$ and a reswitch at $r_2 \neq r_1$. $M(r_1) \cap M(r_2)$ is, as we saw, not empty, because it contains $(\mathbf{a}_1, l_1)$. We assume at first that the labour theory of value holds so that $\bar{\mathbf{p}}(r)$ is constant and $\bar{w} = 1 - r/R$. As the reader will easily verify, $h_{ij}(r_1)$ and $h_{ij}(r_2)$; $i < j$; $0 \leq r_1 < r_2 < R$; will then not intersect, if $j < n+1$, because the line segment $h_{ij}(r)$ shifts downwards to the left as $r$ rises with both $z_i(r)$ and $z_j(r)$ falling. But there will be intersections in the coordinate hyperplanes involving the labour dimension $n+1$. These intersections $P_{1,n+1}(r_1, r_2), \ldots, P_{n,n+1}(r_1, r_2)$ span the $(n-1)$-dimensional simplex $M(r_1) \cap M(r_2)$, and one shows by means of a short calculation that they remain stationary, if $r_2$ changes, given $r_1$, and if and only if relative prices of commodities do not change (see diagram 8). These intersections exist for $0 \leq r_2 < R; r_2 \neq r_1$.

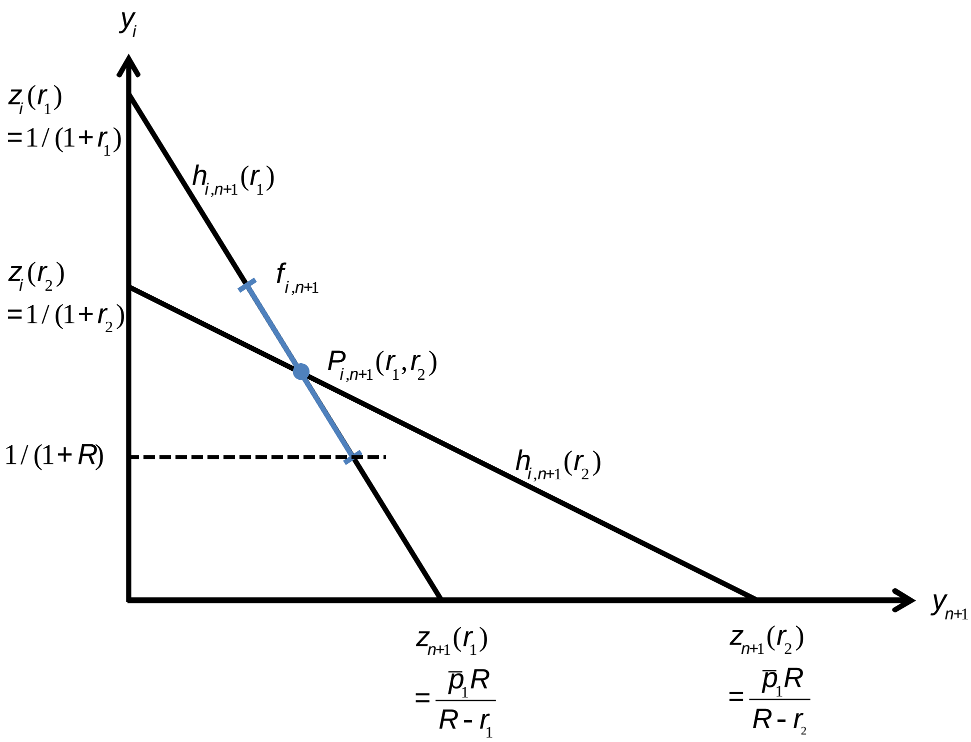


*Diagram 8*: The edges of $M(r_1)$, $h_{i,n+1}(r_1)$, and of $M(r_2)$ $h_{i,n+1}(r_2)$, turn around a stationary point of intersection $P_{i,n+1}(r_1, r_2)$ representing $M(r_1) \cap M(r_2)$ in $H_{i,n+1}$, if the labour theory of value holds. $P_{i,n+1}(r_1, r_2)$ will move along a stretch $f_{i,n+1}$ of $h_{i,n+1}(r_1)$, if prices change.

This means that $M(r_1) \cap M(r_2)$ is constant as a function of $r_2$ and remains the same $(n-1)$-dimensional subset (simplex) of $M(r_1)$, if the labour theory of value holds. As we saw in section 6, it is of $n$-dimensional measure zero, and a method $(\mathbf{a}_0, l_0)$ in $M(r_1)$ will almost surely not be in $M(r_1) \cap M(r_2)$. The methods $(\mathbf{a}_0, l_0)$ in $M(r_1) \cap M(r_2)$ yield the same wage curve as the original technique $(\mathbf{A}, \mathbf{l})$, in that "reswitching" takes place in all $r_2$. One knows that two regular single product systems with wage curves that coincide in an open interval of $0 \le r \le R$ will be identical, but these systems composed of methods $(\mathbf{a}_i, l_i)$; $i = 0, 2, ..., n;$ with $(\mathbf{a}_0, l_0)$ in $M(r_1) \cap M(r_2)$, are not regular. We thus get confirmed what we know: reswitching is unlikely, if the labour theory of value holds.[13]

Next we assume that relative prices change. The movements of $P_{ij}(r_1, r_2)$ in function of $r_2$, given $r_1$, along the edges of $M(r_1)$ in the $n(n-1)/2$ coordinate hyperplanes $H_{1,2}, ..., H_{n,n+1}$ trace stetches $f_{ij}$ on $h_{ij}(r_1)$. Except if $P_{ij}(r_1, r_2)$ happens to coincide with $(\mathbf{a}_1, l_1)$ – which is then not positive – we can connect $P_{ij}(r_1, r_2)$ with its corresponding point $P^*_{ij}(r_1, r_2)$ which is defined as follows. The straight line $g_{ij}(r_1, r_2)$ defined by

$$\lambda P_{ij}(r_1, r_2) + (1-\lambda)(\mathbf{a}_1, l_1); \quad -\infty < \lambda < \infty;$$

will intersect the border of $\mathbb{R}^{n+1}_+$ in one point $P^*_{i*,j*}(r_1, r_2)$ in some hyperplane $H^*_{i*,j*}$; this is the corresponding point. The semi-positive points of $g_{ij}(r_1, r_2)$ are contained in $M(r_1) \cap M(r_2)$. Corresponding points on all $f_{ij}$ span $M(r_1) \cap M(r_2)$; as $r_2$ moves, one obtains the set

$$M^* = \bigcup_{\substack{r_1 \neq r_2 \\ 0 \le r_2 \le R}} M(r_1) \cap M(r_2)$$

introduced in section 6 above, i.e. the set of all potential techniques which give rise to some reswitch, given the switch at $r_1$. $M^*$ is in general not convex, but star-shaped, in that each $(\mathbf{a}_0, l_0)$ is connected to $(\mathbf{a}_1, l_1)$ by a line segment contained in $M^*$ which can be extended to the corresponding points $P_{ij}(r_1, r_2)$ and $P^*_{i*,j*}(r_1, r_2)$. If $(\mathbf{a}_1, l_1)$ is on $h_{ij}(r_1)$, $(a_1, l_1)$ becomes a vertex of $M^*$ (Diagram 9).

[13] But it is not impossible, insofar as the wage curves of two different techniques in $M(r_1) \cap M(r_2)$ coincide in the labour-theory-of-value-case.

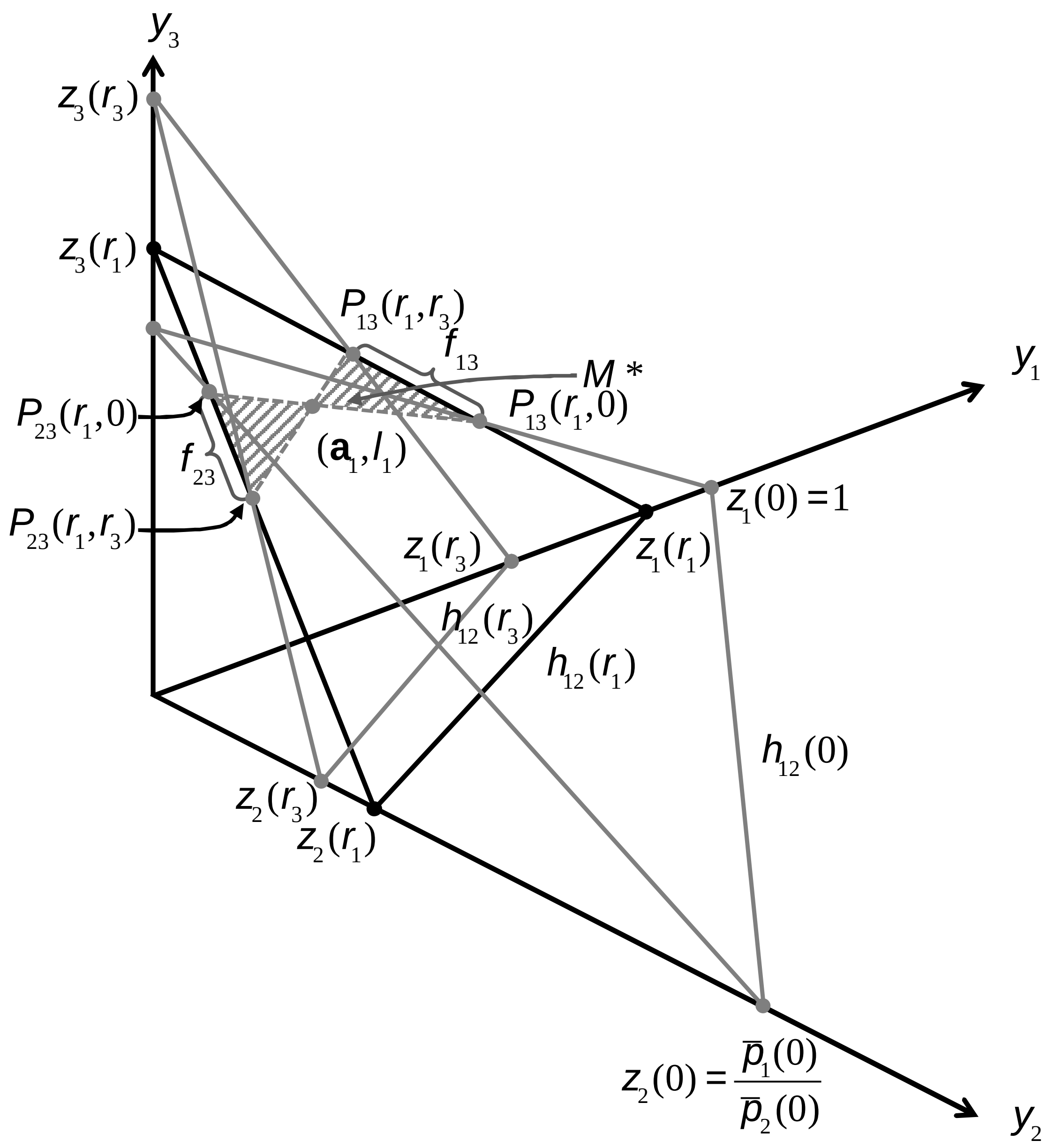


*Diagram 9*: $\boldsymbol{M}^*$, the set of all potential techniques with a switch at $r_1$ and another at some $r_2 \neq r_1$; $0 \le r_2 \le R$; if $n = 2$. $M(r_2)$ is drawn for $r_2 = 0$ and for a large $r_2$ (denoted $\boldsymbol{r}_3$). $M(R)$ is obtained as a limit for $r_2 \to R$.

Because of continuity, each $P_{ij}(r_1,r_2)$ describes a stretch $f_{ij}$ on the corresponding $h_{ij}(r_1)$, as $r_2$ varies; $0 \le r_2 \le R$, $r_2 \neq r_1$; which is connected. This stretch will often be short, as the $P_{ij}(r_1,r_2)$ move little with $r_2$ and not all downwards as diagram 9 illustrates for $n$ = 2. Moreover, $f_{ij}$ is not only short, but empty, if $h_{ij}(r_1)$ and $h_{ij}(r_2)$ do not intersect for any $r_2$; $0 \le r_2 \le R$. This happens in plane $H_{12}$, spanned by $y_1$ and $y_{21}$ in diagram 9. As

$P_{13}(r_1, r_2)$ moves on $f_{13}$ with $r_2$ from $P_{13}(r_1, 0)$ (see diagram 9) to $P_{13}(r_1, r_3)$, the corresponding points $P_{23}(r_1, r_2)$ move on $f_{23}$ in the opposite direction from $P_{23}(r_1, 0)$ (see diagram 9) to $P_{23}(r_1, r_3)$, if $(\mathbf{a}_i, l_i) > 0$. This characteristic retrograde movement, necessary for a positive probability of reswitching, can also be observed in higher dimensions by constructing a three-dimensional subspace in $\mathbb{R}^{n+1}$.

Let $M^{**}$ be the convex hull of $M^*$. Clearly, $M^{**}$ is the convex hull of the $f_{ij}$; $1 \le i < j \le n+1$. $\overline{f}_{ij}$ is the length of $f_{ij}$; $\overline{h}_{ij}$ the length of $h_{ij}$. We shall say that $(\mathbf{A}, \mathbf{l})$ is bounded in $i, j$, if there is $\alpha_{ij}$, $0 \le \alpha_{ij} < 1$ with $\overline{f}_{ij} / \overline{h}_{ij} \le \alpha_{ij}$. If $f_{ij}$ is empty, $\alpha_{ij} = 0$. To assume $\alpha_{ij} < 1$ is easy to motivate for $i = 1$ (the industry where the switches take place) and $j = n+1$ (the labour dimension). Consider the movement of $P_{1,n+1}(r_1, r_2)$ as a function of $r_2$, given $r_1$, in diagram 8. The edge of $M(r_1)$, $h_{1,n+1}(r_1)$, then is spanned by the fixed endpoints $z_1(r_1) = 1/(1+r_1)$ and $z_{n+1}(r_1) = R\overline{p}_1 / (R - r_1)$, while $h_{1,n+1}(r_2)$ moves with its endpoints $z_1(r_2)$ and $z_{n+1}(r_2) = R\overline{p}_1 / (R - r_2)$ according to the level of $r_2$. We begin to illustrate a possible such movement with $0 < r_2 < r_1 < R$ and $r_2$ near zero. We are close to the intersection of $h_{1,n+1}(r_1)$ and $h_{1,n+1}(r_2)$, given by the point $P_{ij}(r_1, r_2)$ which would result in the labour value case. Only because $\overline{p}_1(r)$ deviates from the labour value, $P_{1,n+1}(r_1, r_2)$ now moves downwards. For $r_2 \to r_1$, $h_{1,n+1}(r_2)$ will coincide with $h_{1,n+1}(r_1)$, but only one point of $h_{1,n+1}(r_1)$ will, as a limit point, belong to $M^*$, denoted by $P_{1,n+1}(r_1, r_1)$; the corresponding method does not lead to reswitching, properly speaking, but to coinciding wage curves. Next, on the abscissa, $z_1(r_2) = 1/(1+r_2)$ will move towards $1/(1+R)$, while $z_{n+1}(r_2) = R\overline{p}_1 / (R - r_2)$ will move to infinity, $\overline{p}_1(r_2)$ remaining positive and tending to $\overline{p}_1(R)$. Hence the abscissa of $P_{1,n+1}(r_1, r_2)$ remains above $1/(1+R)$ and $\overline{f}_{1,n+1} < \overline{h}_{1,n+1}$. Other scenarios, depending on the deviation of $\overline{p}_1(r_2)$ from the labour value, are possible, but they mostly also lead to the conclusion that $\overline{f}_{1,n+1} < \overline{h}_{1,n+1}$. Similar results hold for $f_{2,n+1}, \ldots, f_{n,n+1}$.

Complexity increases for the cases $i < j < n+1$; we limit the analysis to the statement of a condition necessary to get $\overline{f} = \overline{h}$: There must be $r_2$ and $r_3$ in $[0, R]$ such that $h_{ij}(r_1)$ and $h_{ij}(r_2)$ intersect on the abscissa

$$\frac{\overline{p}_1(r_1)}{\overline{p}_1(r_2)} = \frac{1+r_1}{1+r_2} \frac{\overline{p}_i(r_1)}{\overline{p}_i(r_2)}$$

and on the ordinate

$$\frac{\bar{p}_1(r_1)}{\bar{p}_1(r_3)} = \frac{1+r_1}{1+r_3}\frac{\bar{p}_i(r_1)}{\bar{p}_i(r_3)}.$$

These conditions are not impossible to fulfil simultaneously, but they require specific deviations from labour values in the form of changes in the direction of relative prices. For instance, we have to begin with an intersection on the abscissa, $r_2 < r_1$ and $z_i(r_2) = z_i(r_1)$. As $r_2$ rises, we return to $z(r_2) = z_i(r_1)$ at $r_2 = r_1$, so that $z(r_2)$ falls, then rises but must fall again, for the point of intersection must move to the ordinate until $z_j(r_2) = z_j(r_1)$. This also involves another retrograde movement. We must have had $z_j(r_2) = z_j(r_1)$ already earlier, when $r_2 = r_1$. Oscillations of both $z_i$ and $z_j$ thus are necessary to bring about an $f_{ij}$ which coincides with $h_{ij}$. In consequence, one expects $0 \le \alpha_{ij} < 1$ for most, if not for all $i, j$.[14]

$M^*(r_1)$, the set of potential techniques giving rise to reswitching at some $r_2 \in [0, R]$, $r_2 \ne r_1$, is by definition contained in $M(r_1)$ and is truly a partial set, since e.g. points $\xi \mathbf{e}_1$, with $0 < \xi < 1/(1+R)$ are never in $M(r_1)$, as can be seen by considering $P_{1,\,n+1}(r_1, r_2)$.[15] The point $\xi \mathbf{e}_1$ will have an open neighbourhood in $M(r_1)$ where the potential techniques do not give rise to reswitching. Hence $\mu(M^*) < \mu\left(M(r_1)\right)$. Now $M(r_1)$ is an $n$-dimensional simplex in $\mathbb{R}^{n+1}$ spanned by $n+1$ vectors $z_1(r_1)\mathbf{e}_1, ..., z_{n+1}(r_1)\mathbf{e}_{n+1}$. The volume of a simplex increases with the distance between the vertices spanning it. There must be a smaller simplex $M^{***}$, geometrically similar to $M(r_1)$, given by $\gamma z_1(r_1)\mathbf{e}_1, ..., \gamma z_{n+1}(r_1)\mathbf{e}_{n+1}$; $0 < \gamma < 1$; such that $\mu(M^*) = \mu(M^{***})$. We shall say that $(\mathbf{A}, l)$ is bounded by $\gamma$, or, simply that it is bounded.

Much could be investigated and said about the relationships between $M^{**}$, the convex hull of $M^*$, which is spanned by the $f_{ij}$, about the lengths $\overline{f_{ij}}$ of the $f_{ij}$, and their

[14] Perhaps the following visualisation may help. Think of $h_{ij}(r_1)$ as of a fixed rod and of $h_{ij}(r_2)$ as of a moving lever, with a turning point $P$ corresponding to the labour-theory-of-value-case somewhere in the middle of the rod. Lever and rod coincide for $h_{ij}(r_1)$. Lift the lever and turn it slightly counter clockwise to represent the initial movement ($r_2$ small). As $r_2$ rises, lower the lever so that an intersection of rod and lever arises on the ordinate. The intersection then descends to the right as the lever is lowered. But the lever now must be turned clockwise to get the superposition of rod and lever at $r_2 = r_1$. Turning it further and raising it will cause the intersection to move downwards to the abscissa, implying a prior retrograde movement there.

[15] See preceding footnote.

relationship with $\mu(M^{*})$ and $\mu(M^{**})$, and about geometrical conditions defining the relationship between the $\alpha_{ij}$ and $\gamma$, but not much would be gained for our economic understanding: It is clear that $\gamma$ will increase and diminish with the $f_{ij}$ and $\alpha_{ij}$ in the comparison of systems. $M^{*}$ is something like an irregular contraction of $M(r_1)$, represented in a rough manner by $M^{***}$ and by $\gamma$. The volume of the simplex $M(r_1)$ needed to calculate the probability depends explicitly only on the edges $h_{1,\,n+1},\ldots,h_{n,\,n+1}$ connecting the vertices $z_1\mathbf{e}_1,\ldots,z_n\mathbf{e}_n$. These become shorter, if we replace $h_{i,\,n+1}$ by $f_{i,\,n+1}$; $i=1,\ldots,n$; and $\gamma$ reflects this contraction as a kind of average. We found that the relative prices in the exchanges between commodities and labour (prices in terms of labour commanded or in terms of the wage rate) played the main role for the explanation of the positive probability of reswitching, as soon as these prices begin to deviate from labour values, and all relative prices can be expressed in terms of $p_1/w,\ldots,p_n/w$, for $p_i/p_j=(p_i/w)(p_j/w)$. This also may justify our definition of boundedness in terms of the vertices $z_i\mathbf{e}_i$, connected by the edges $h_{1,\,n+1},\ldots,h_{n,\,n+1}$.

Hence we come directly to the conclusion. Let an euclidian $n$-dimensional coordinate system be given in the $n$-dimensional hyperplane containing $M(r_1)$ and let the vertices $z_i(r_1)\mathbf{e}_i$ of $M(r_1)$ in these coordinates be expressed by $n+1$ $n$-vectors $\mathbf{v}_i$. The $n$-dimensional volume $V$ of $M(r_1)$ then is given by

$$V=\frac{1}{n!}\left|\det(v_1-v_{n+1},\ldots,v_n-v_{n+1})\right|,$$

the volume $V^{***}$ of $M^{***}$ is given by

$$V^{***}=\frac{1}{n!}\left|\det(\gamma v_1-\gamma v_{n+1},\ldots,\gamma v_n-\gamma v_{n+1})\right|.$$

The likelihood of reswitching

$$\pi=\mu(M^{*})/\left(\mu M(r_1)\right)$$

therefore can be expressed for systems bounded by $\gamma$ as

$$V^{***}/V=\gamma^{n}.$$

Obviously $\pi$ tends to zero as $n\to\infty$.

The question immediately arises as to what systems are bounded in this sense, besides those for which the labour theory of value holds approximately. One could argue, against our result, that higher dimensions mean more possibilities for prices to oscillate, but, on the other hand, higher dimensions mean more averaging out of price movements (if the matrix is not sparse), as became clear in the discussion about the approximate production function. Finally, if we directly and more simply assume a large spectral gap, again the movements of relative prices, hence the $f_{ij}$, $\alpha_{ij}$ and ultimately $\gamma$ tend to be bounded and to remain so, as $n$ increases.

The situation is quite different from that encountered in the case of Wicksell effects. There, the likelihood for Wicksell effects of some kind is equal to one for all $n \geq 2$, only the magnitude is likely to diminish because of the laws of averages. More would have to be said about the structure of the system in order to define the likelihood of neoclassical Wicksell effects or their opposite. If one starts from the consideration of the standard wage curve, both seem about equally likely, and, though in different degrees, both are a nuisance for neoclassical and keynesian theories.

Petri (2011, p. 407) following a suggestion by Garegnani, draws a demand curve for capital (with the intensity of capital $k$ on the ordinate and the rate of profit on the abscissa) exhibiting a saw-like movement upwards. The rising stretches are due to anti-neoclassical Wicksell effects, the vertical drops to neoclassical switches (no reverse capital deepening). In the drawing, the anti-neoclassical tendency of the Wicksell effects prevails over the neoclassical reduction of $k$ in that the curve as a whole has a rising tendency; one might speak of a series of capital reversals. We indeed found a reason in section 6 letting it appear probable that Wicksell effects will come in groups. It seems less likely that such a rising tendency will extend over a larger interval, in that times and again the rise due to repeated anti-neoclassical Wicksell effects prevails over the fall of $k$ due to neoclassical switches, but Zambelli's results do imply demand curves for capital with shorter intervals of this kind.

The analysis of the likelihood of reswitching here presented concerns only the potential alternatives in one industry. Its application to the choice of technique, with a full spectrum of alternative methods in each industry, first yields the insight that reswitching as mathematically expressed in this Appendix also applies to reverse capital deepening. This, we repeat, at least according to our preferred definition, has nothing to do with capital reversals due to Wicksell effects. It takes place, if two wage curves of systems differing in the method employed in one industry intersect twice at $r_1$ and $r_2$, with one switchpoint at the higher rate of profit being on the envelope, and the other dominated by the wage curve of a third technique which switches at some $r_3$ between $r_1$ and $r_2$ (see diagram 4 above). Reverse capital deepening was encountered more often in Han and Schefold (2006) than the formula $\pi = \gamma^n$ suggests, because there were more possibilities for reverse capital deepening: one switch takes place below the envelope.

The analysis for the likelihood of reswitching and reverse capital deepening applicable to the entire complex of wage curves derived from a full spectrum of techniques remains to be worked out. It will still turn out that reswitching is rare, especially, for large systems, although more technical alternatives come up, because the number of wage curves of techniques appearing on the envelope increases more slowly (with the logarithm of $s$, if $s$ is the number of alternative techniques, according to Schefold 2013b). So we have several independent reasons to argue why the paradoxes of capital are less visible or less important in large systems, but some of these considerations entail new criticisms of the traditional theories.

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