

Piero Sraffas Theorie der Kuppelproduktion,
des Kapitals und der Rente
(Mr. Sraffa on Joint Production)

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Meinen Eltern

Vorwort

"It is no wonder that this book* took a long time to write. It will not be read quickly**". So begann Joan Robinson vor zehn Jahren ihre Rezension von Piero Sraffas "Production of Commodities by Means of Commodities". Und in der Tat: trotz einer Flut von Artikeln (s. Harcourts Uebersicht***) ist das Werk noch immer erst partiell rezipiert. Nicht nur ist die Kritik an der Neoklassik noch nicht wirklich aus dem Stadium des "Präludiums" hervorgewachsen, Sraffas Buch selbst ist erst zum Teil durchdiskutiert. Ausser Manara**** dessen Verdienst mehr in seinen Fragen als seinen Antworten liegt, hat unseres Wissens noch kein Autor sich ausführlich mit der Kuppelproduktion als dem Hauptteil des Buchs auseinandergesetzt.

Dies ist umso erstaunlicher, als Sraffa sich mit einem zentralen Thema der grossen Oekonomen seit der Klassik beschäftigt, nämlich dem Kapitalbegriff. Nach Sraffa ist in der Tradition von Torrens, Ricardo usw.***** fixes Kapital nur als Kuppelprodukt zu verstehen. Demgegenüber hat die Diskussion in den Fachzeitschriften sich bisher auf Modelle mit ausschliesslich zirkulierendem Kapital beschränkt.

* Piero Sraffa, Production of Commodities by Means of Commodities (/11/)

** Joan Robinson (/33/). *** (/21/). **** (/6/).

***** (Sraffa, /11/. Appendix D).

Die vorliegende Dissertation soll dazu beitragen, die Lücke zu schliessen. Zu danken habe ich für wichtige Gespräche beim Zustandekommen dieses Buchs den in der englischen Einleitung Genannten, vor allem aber Herrn Professor Dr. G. Bombach, ohne dessen verständnisvolle Beratung ich nie soweit gelangt wäre. Ferner gilt mein Dank für Forschungsstipendien dem Schweizerischen Nationalfonds (1969) und King's College, Cambridge (1970).

Basel, im Mai 1971

Bertram Schefold

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Zur Einführung

1) Piero Sraffa (geboren 1898) trat zum ersten Mal 1925 mit einem Artikel "Sulle relazioni fra costo e quantità prodotta*" (über das Verhältnis von Ertrag und Kosten) als Kritiker der Marshall'schen Theorie des partiellen Gleichgewichts hervor. Ausgangspunkt seiner Kritik war die damals weithin ungenügend reflektierte Unterscheidung zwischen steigenden, fallenden und konstanten Skalenerträgen. Entgegen der Meinung, wonach die Schwierigkeit, diese Unterscheidung in der Realität zu verifizieren, auf das Ungenügen der Statistiken zurückzuführen sei, griff Sraffa das "fundamentum divisionis" selbst an: ob in einer bestimmten Industrie steigende, fallende oder konstante Erträge vorliegen, hängt vom Standpunkt des Beobachters ab. Je nachdem, ob man die kurze oder die lange Periode, den ganzen Industriezweig oder nur eine einzelne Unternehmung in Betracht zieht, kann der eine oder der andere Fall eintreten. Außerdem wird allgemein die Idee der Abhängigkeit der produzierten Warenmenge von den Produktionskosten bei vollkommener Konkurrenz ** durch die Erfahrung nicht suggeriert***; nach Sraffa wurde diese Abhangigkeit vielmehr erst von den Nutzentheoretikern postuliert, die der fallenden Nachfragekurve gewissermassen aus Symmetriegründen eine steigende Angebotskurve gegenüber stellen wollten. Dies geschah in Opposition zu den Klassikern, die mit Ricardo für die meisten Waren konstante Skalenerträge annahmen. Von den Neoklassikern hatten sich einige konstante Skalenerträge überhaupt nur als Sich-aufheben der gegenläufigen Tendenzen von steigenden und fallenden Zahlerträgen vorstellen können. Sraffa dagegen bemerkte, wieviel natürlicher es ist, von der

* / 37 / ** d.h. im langfristigen, stationären Gleichgewicht
*** loc.cit. Seite 279

Abwesenheit beider Tendenzen auf konstante Skalenerträge, die dann nicht die Ausnahme, sondern die Regel sind, zu schliessen. In einer langen und komplizierten Argumentation versucht er, ausserdem nachzuweisen, dass die Theorie der partiellen Gleichgewichte in einem statischen System unter vollkommener Konkurrenz bis auf bestimmte seltene Ausnahmen überhaupt nur mit konstanten Skalenerträgen verträglich ist, sodass die Preise nur von der Kostenseite, nicht von der Nachfrageseite her bestimmt erscheinen. Seine Argumentation beruht wesentlich darauf, dass die Interdependenz der Märkte bei variablen Skalenerträgen mit der für die Theorie des partiellen Gleichgewichts notwendigen Unabhängigkeit von Angebots- und Nachfragekurven im Widerspruch steht. 1926, in seinem Artikel im Economic Journal ("The Laws of Return under Competitive Conditions") folgte er weiter, dass (wie auch die Erfahrung zeige) das Verkaufsvolumen miteinander konkurrierender Betriebe nicht durch den Kostenverlauf, sondern die Nachfrage gegeben sei. Wenn sich aber jedes einzelne Unternehmen einer individuellen Nachfragesituation gegenüber sieht, so verkauft jedes Unternehmen in einem besonderen Markt, in dem es als Monopolist oder Oligopolist auftritt.

Die ökonomische Forschung wandte sich bekanntlich in der Folge* der Theorie der unvollkommenen Konkurrenz zu, ohne den Marshall'schen Ansatz grundsätzlich aufzugeben.

Sraffa jedoch unternahm seit spätestens 1928** etwas radikal anderes.

*Napoleoni, Grundzüge der modernen ökonomischen Theorien S. 45 ff (/27/).

** Vorwort "Production of Commodities by Means of Commodities", (/11/).

Er untersuchte, wie die Produktionspreise ohne Rücksicht auf Skalenerträge und Nachfragekurven gerade aus der Interdependenz der Märkte erklärt werden können und fügte also die Elemente seiner Kritik an der vorherrschenden Theorie neu zusammen, um mit einer vertieften Kritik auch eine alternative Theorie zu konstruieren.

Zur Durchführung dieses Programms war ein völliges Umdenken erforderlich. Wahrscheinlich empfing Sraffa die entscheidenden Anregungen von der Lektüre der Klassiker, insbesondere Ricardos, dessen Herausgeber er auf Anregung von Keynes bald darauf wurde. In jedem Fall sind in Sraffas Vorwort zur Ricardo-Ausgabe die Grundgedanken zu dem, was später "Production of Commodities by Means of Commodities" wurde, zu finden. Da andere Veröffentlichungen Sraffas aus jener Zeit nicht vorliegen, wollen wir auf dieses Vorwort* zur Einführung in die Gedankengänge von "Production of Commodities by Means of Commodities" etwas näher eingehen.

- 2) Ricardo betrachtet bekanntlich eine geschlossene Volkswirtschaft mit uniformer Lohn- und Profitrate und fragt nach der Verteilung des Sozialprodukts auf die Klassen. Wodurch diese Verteilung bei ihm bestimmt ist (Arbeitsproduktivität auf dem marginalen Land, usw.), wollen wir hier offenlassen - uns interessieren viel mehr die Preise bei verschiedenen, gegebenen Verteilungen, genauer, die "natürlichen Preise". Der natürliche Preis einer Ware ist gleich der

* in Ricardos "Principles" (/31/)

Summe der Werte der in die Herstellung der Ware eingehenden Produktionsmittel einschliesslich der Arbeit (die auch einen durch die Lebensbedürfnisse der Arbeiter bestimmten Wert besitzt), zuzüglich dem bei der Herstellung erzielten und durch die vorgegebene Profitrate determinierten Profit, sowie gegebenenfalls noch zuzüglich der Rente. Der natürliche Preis ist also der durch die ~Profit und Lohnrate uniformisierende-Konkurrenz in der langen Periode hergestellte Durchschnittspreis in einem statischen System. Ihm steht der von Angebot und Nachfrage bestimmte Marktpreis der kurzen Periode gegenüber, der uns hier nichts weiter angeht.

1816, während der Arbeit an der ersten Auflage der *Principles*, machte Ricardo die überraschende Entdeckung, dass unter den Bedingungen eines solchen Systems (in dem der natürliche Preis jeder Ware festgelegt ist) die Preise von Waren, die "hauptsächlich mit Hilfe von Maschinerie und fixem Kapital hergestellt werden" infolge einer Lohnsteigerung relativ zu den Preisen hauptsächlich durch direkte Arbeit hergestellter Waren fallen können*. Eine Steigerung der Löhne hat also nicht unbedingt eine Steigerung aller Preise (ausgedrückt etwa in Gold) zur Folge. Wie sind Änderungen der Verteilung zwischen den Klassen der Lohnarbeiter, der Kapitalisten und Landbesitzer zu beschreiben, wenn das zu verteilende Produkt mit der Verteilung dem Wert nach ändert?

Ricardos erste Antwort bestand in einem Versuch, das Problem zu umgehen**. Man nehme an, in allen Zweigen der Produktion wird

* loc. cit. Seite XVI

** loc. cit. Seite XXXI, Sraffa, *Production of Commodities by Means of Commodities*, Appendix D1

Korn, das heisst Arbeit, direkt oder indirekt als Produktionsmittel verwendet, während in die Produktion von Korn selbst nur Korn (als einziges Subsistenzmittel der Arbeiter sowohl wie als Saatgut) eingeht. Wenn wir von Rente absehen, so bestimmt der Kornbedarf der Arbeiter zusammen mit der physischen Reproduktionsrate des Korns den Profit in der Landwirtschaft und damit in allen anderen Produktionszweigen. Die Verteilung des Volkseinkommens erscheint hier zugleich determiniert und anschaulich beschrieben. Auf Kosten einer starken, jedoch im England Ricardos nicht völlig abwegigen Vereinfachung, ist die Profitrate ausgedrückt als der Quotient zweier gleichartiger Größen: der Profit und das Kapital sind beide in Korn gemessen, und alle andern Profitraten haben sich nach dieser einen zu richten.

Was aber, wenn in Zähler und Nenner der Profitrate eine heterogene Sammlung verschiedenartiger Waren mit ihren Preisen auftritt? Eine Änderung der Profitrate induziert Änderungen der Preise und umgekehrt. Der Wert des Gegebenen, des Kapitals, sowohl wie der des zu Verteilenden, des Volkseinkommens, fluktuiert.

Ricardo unternahm es nun (und nach Sraffa liess ihn die Frage sein Leben lang nicht los), eine Ware zu suchen, deren Preis bei einer Änderung der Verteilung invariant bleibt. Für ihn hatte die Kenntnis einer solchen Standard-Ware nicht nur den grossen, aber vielleicht bloss akademischen, Vorteil, dass sie die Änderungen der Verteilung exakt zu beschreiben erlaubte, sondern sie hatte auch das eminente praktische Interesse, als Maßstab absoluten Werts verschiedener Kapitale dienen zu können*. Er scheint sogar gehofft zu haben, einen

* vgl. Brief an McCulloch (loc. cit. Seite xlix)

Begriff absoluten Werts finden zu können*, der auf Verbesserungen der Produktionsmethoden reagiert, dagegen bei Verteilungsänderungen konstant bleibt. Wie im ersten Kapitel der "Principles" dargelegt wurde, wusste Ricardo, dass eine solche Standard-Ware nicht in Wirklichkeit existieren kann und ein Maßstab solchen absoluten Werts noch weniger. Aber er glaubte, wenigstens eine gute Annäherung an die Standard-Ware gefunden zu haben, wenn als Maß ein Produkt genommen wurde, bei dem das Verhältnis von aufgewandter Arbeit zu verzehrtem Kapital dem Durchschnitt in der ganzen Volkswirtschaft entsprach. Der Wert einer solchen Durchschnittsware wird sich mit der Verteilung wenig ändern. Außerdem sind in diesem Standard sowohl der Durchschnittspreis aller Güter, wie ihr aggregierter Wert invariant gegen Änderungen der Verteilung**. Die beiden angeführten Probleme hängen insofern zusammen. Was die Praxis angeht, so schien Gold ihm als ein solcher Standard durchaus infrage zu kommen***.

3) Es ist Sraffas grosses Verdienst, die Grundgedanken Ricardos zu einem kohärenten logischen System, das zur Preistheorie der Marginalisten eine echte Alternative bildet, ausgearbeitet zu haben. Wie weit er dabei Ricardo entnommene Ideen weiterspann oder wie weit er Ricardo verstehen konnte, weil er diese Ideen selbst unabhängig entwickelt hatte, ist gleichgültig. Als Resultat liegt jedenfalls in Production of Commodities by Means of Commodities ein Werk vor,

*loc. cit. Seite xlvii

** loc. cit. Seite xliv

*** loc. cit. Seite 46

das neben der Neoklassik Marshalls und der Marxistischen Oekonomie eine neue dritte Fortsetzung Ricardianischer Ideen darstellt, die zu den beiden andern in fruchtbarem Gegensatz steht*. Man darf wohl hinzufügen, dass die schöne klassische Form, in die Sraffa seine Abhandlung gekleidet hat, der Prätention solchen Inhalts entspricht.

Die in diesem Zusammenhang gehörigen weiteren geistesgeschichtlichen, methodologischen und auch empirischen Betrachtungen müssen wir uns leider versagen. Unser Ziel ist ja nur, für Sraffas Behandlung der Kuppelproduktion, den technisch schwierigsten Teil seines Buchs, eine mathematische Ausarbeitung zu geben. Zum Verständnis des mathematisch

* Sraffas Opposition zur Neoklassik wird unten noch weiter begründet werden. Sie betrifft nicht alle Züge der neoklassischen Theorie. Ebenso steht Sraffa auch zur marxistischen Oekonomie nicht in volliger Opposition: sein Ansatz ist formal ähnlich dem von Marx in "Das Kapital", Band III, verwendeten. Daher wird Sraffa von Joan Robinson u.a. zur Erklärung spezieller Fragen bei Marx, insbesondere des Transformationsproblems (s.h. z.B. Joan Robinson "Value and Price", (/34/)) herangezogen . Production of Commodities by Means of Commodities ignoriert aber sowohl die Ausbeutungslehre wie die dialektische Methode.

gehaltenen Texts ist eine vorhergehende Lektüre des ganzen Buchs
Production of Commodities by Means of Commodities unerlässlich.

Wir benützen den Rest dieser Einführung, um zu zeigen, wie die Verbindung von Ricardos Grundgedanken zu der mathematischen Struktur, die im vorliegenden Buch diskutiert wird, herzustellen ist. Die folgenden Abschnitte 4-9 unterscheiden sich in der Darstellung von der Sraffas, fügen jedoch inhaltlich nur wenig Neues hinzu.

4) Heute, nach Leontief und von Neumann, ist es leicht geworden, Ricardos Modell einer geschlossenen Volkswirtschaft ohne Kuppelproduktion mathematisch auszudrücken. Nehmen wir an, es gibt n Waren $j=1, \dots, n$ und die Industrien des Landes gliedern sich entsprechend in n Sektoren $i=1, \dots, n$, die je das Gut i herstellen. Die i -te Industrie verwendet pro Produktionsperiode die Quantitäten a_i^1, \dots, a_i^n als Inputs sowie ℓ_i Arbeitsstunden, um eine Einheit des Guts i herzustellen; symbolisch:

$$(a_i^1, \dots, a_i^n, \ell_i) \longrightarrow (0, \dots, 0, 1, 0, \dots, 0),$$

oder für alle Industrien zusammen:

$$(a_1^1, \dots, a_n^n, \ell_1) \longrightarrow (1, \dots, 0)$$

$$(a_1^1, \dots, a_n^n, \ell_n) \longrightarrow (0, \dots, 1).$$

Die Bedingung der Reproduktion des Wirtschaftssystems als ganzem erfordert, dass von jedem einzelnen Gut mindestens so viel produziert wird, wie in der Produktion aller anderen Güter aufgezehrt wird: wenn von jedem Gut j gerade eine Einheit hergestellt wird, so muss $a_1^j + \dots + a_n^j \leq 1, j=1, \dots, n$, sein. Wir normieren $\ell_1 + \dots + \ell_n = 1$.

Wir berechnen nun die Arbeitswerte. Zur Produktion des Gutes i wurden im laufenden Jahr mit Hilfe der Arbeit ℓ_i die Gütermengen $a_{i1}^1, \dots, a_{in}^n$ aufgewandt. In einer Einheit des Gutes i steckt also jedenfalls die direkte Arbeit ℓ_i^1 aus dem laufenden Jahr, ebenso aber auch die indirekte Arbeit aus früheren Jahren, die sich in den Produktionsmitteln $a_{i1}^2, \dots, a_{in}^n$ verkörpert. Wieviel beträgt diese indirekte Arbeit? Nehmen wir a_i^1 . a_i^1 ist der a_i^1 -te Teil der Gesamtproduktion von Gut 1 im vorigen Jahr und enthält somit $a_i^1 \ell_i^1$ an direkter Arbeit aus dem vorigen Jahr, sowie die in den Produktionsmitteln $a_i^2(a_{i1}^1, a_i^2)$ verkörperte indirekte Arbeit. Diese kann weiter aufgelöst werden in direkte Arbeit aus dem vorvorigen Jahr und indirekte, in den Produktionsmitteln verkörperte Arbeit aus noch früheren Jahren.

Wenn wir die in der Produktion des Gutes i im laufenden Jahr aufgewandte direkte Arbeit ℓ_i^1 mit $L_i^{(1)}$ bezeichnen, die in i indirekt eingehende Arbeit aus dem Vorjahr mit $L_i^{(2)}$, die aus dem Vorvorjahr mit $L_i^{(3)}$ usw., so haben wir

$$L_i^{(1)} = \ell_i^1$$

$$L_i^{(2)} = a_i^1 \ell_i^1 + \dots + a_i^n \ell_n$$

$$L_i^{(3)} = a_i^1 (a_{i1}^1 \ell_i^1 + \dots + a_{in}^n \ell_n) + \dots + a_i^n (a_{i1}^1 \ell_i^1 + \dots + a_{in}^n \ell_n)$$

-----,

sodass sich die ganze in einer Einheit des Gutes i verkörperte Arbeit L_i auflösen lässt in

$$L_i = L_i^{(1)} + L_i^{(2)} + L_i^{(3)} + \dots$$

Unter Benützung der Matrixschreibweise kann man diese Formeln vereinfachen. Für die Inputmatrix der (a_{ij}^1) schreiben wir A ,

$$(a_i) = \begin{bmatrix} a_{11}^i & \dots & a_{1n}^i \\ a_{21}^i & \dots & a_{2n}^i \\ \vdots & \ddots & \vdots \\ a_{n1}^i & \dots & a_{nn}^i \end{bmatrix} = A,$$

die Outputmatrix ist die Einheitsmatrix I.

$\ell = \begin{bmatrix} \ell_1 \\ \vdots \\ \ell_n \end{bmatrix}$ ist der Vektor der direkten Arbeit, sodass sich die Produktion symbolisch so ausdrückt:

$$(A, \ell) \longrightarrow I.$$

Mit L als dem Spaltenvektor der direkten und indirekten Arbeit, $L^{(0)}$ als dem Spaltenvektor der vor k Jahren aufgewandten Arbeit, erhalten wir, wie man leicht überlegt:

$$L^{(0)} = \ell$$

$$L^{(1)} = AL$$

$$L^{(2)} = AAL = A^2\ell$$

Andererseits ist es klar, dass die totale in einem Gut i verkörperter Arbeit L_i der Summe der direkten in i eingehenden Arbeit ℓ_i plus der gesamten in den Inputs $a_{1i}^i, \dots, a_{ni}^i$ verkörperten indirekten Arbeit, die gleich $a_{1i}^1L_1 + \dots + a_{ni}^iL_n$ sein muss, entspricht. Also (in Vektorschreibweise):

$$L = \ell + AL,$$

oder

$$L = (I - A)^{-1}\ell,$$

sodass wir die Identität

$$L = (I - A)^{-1}\ell = \ell + AL + A^2\ell + \dots$$

gewonnen haben.

Diese Identität lässt sich auch streng mathematisch beweisen,

vorausgesetzt, dass die Bedingung der Reproduktion $\sum_{i=1}^n a_i^j \leq 1$ für alle j erfüllt ist*.

5) Nun wenden wir uns von den technischen zu den ökonomischen Produktionsbeziehungen.

Sei $p = \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix}$ der Preisvektor der n Waren,

sei w die Lohn-, r die Profitrate. Dann lauten die Preisgleichungen des Ricardo-Sraffaschen Systems:

$$(1+r) A p + w l = p,$$

oder ausgeschrieben:

$$(1+r)(a_1^1 p_1 + \dots + a_1^n p_n) + w l_1 = p_1$$

— — — —

$$(1+r)(a_n^1 p_1 + \dots + a_n^n p_n) + w l_n = p_n.$$

Dieses Gleichungssystem in den $n+2$ Unbekannten p_1, \dots, p_n, w, r hat zwei Freiheitsgrade.

Den ersten interpretieren wir als die Verteilung. Diese ist entweder durch eine irgendwie vorgegebene Profitrate determiniert analog Ricardos Korn-Korn-Modell oder sie wird durch einen fixen Reallohn aus den Gütern b^1, \dots, b^n durch eine zusätzliche Gleichung $b^1 p_1 + \dots + b^n p_n = w$ gegeben.

Bei vorgegebenem w oder r hat das verbleibende Gleichungssystem noch $n+1$ Unbekannte und bestimmt die relativen Preise. Um von den relativen Preisen zu den absoluten zu kommen, muss ein geeigneter

* s.h. unten §§ 4,6.

Standard definiert werden. Ricardo hatte gewisse, in mancher Hinsicht zuweit gehende, Postulate für einen solchen Standard aufgestellt. - Straffa zeigt, durch welche Konstruktion diese Postulate wenigstens zum Teil erfüllt werden können. Die den Standard definierende Gleichung kann vorläufig zum Beispiel $p_i = 1$ für irgendein i lauten.

6) Wir versuchen uns zunächst an einer Diskussion der absoluten Preise, ohne uns auf einen bestimmten Standard festzulegen. Was immer er sei: wenn ein Preisstandard und w durch je eine zusätzliche Gleichung gegeben sind, so lassen sich die Preise aus

$$(1+r) A p + w l = p$$

gemäß

$$p = w (I - (1+r) A)^{-1} l$$

berechnen und die Profitrate ist determiniert.

w ist in einem ökonomischen System zu jedem Zeitpunkt in Wirklichkeit ein festes Datum. Um den Charakter der hier ins Auge gefassten "natürlichen Preise" zu verstehen, ändern wir die Verteilung in Gedanken, indem wir w ändern und beobachten die Auswirkung dieser virtuellen Verschiebungen ^{auf} r und p .

Der Lohn w wird für einen bestimmten Standard sein Maximum w_m erreicht haben, wenn der Profit, also r , gleich Null ist, d. h. wenn das gesamte Volkseinkommen den Arbeitern zufällt:

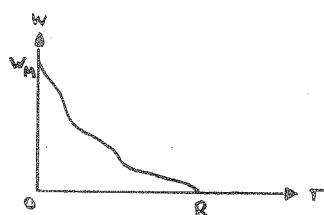
$$A p + w_m l = p$$

Wird der Lohn gesenkt, so steigen die Profite bis zu einer maximalen

Profitrate R , bei der der Lohn gänzlich verschwindet. Diese maximale Profitrate ist offensichtlich unabhängig vom gewählten Standard und sie ist, darin liegt die eigentliche Entdeckung**, endlich. Sie ist Lösung der Gleichung

$$(1+R) A_p = p.$$

Dazwischen hängen w und r funktionell zusammen. Es liegt nahe zu vermuten, und kann (wenn keine Kuppelproduktion vorliegt) streng bewiesen * werden, dass r bei jedem Standard monoton steigt, wenn w fällt:



Diese Relation ist die Grundlage aller folgenden Überlegungen. — Bei der entgegengesetzten Bewegung von w und r ändern sich die Preise kontinuierlich. Einerseits folgt dies direkt aus

$$p = w (I - (1+r) A)^{-1} l,$$

Andererseits lässt sich dieser Ausdruck wie oben in eine Reihe entwickeln

$$p = w (I + (1+r) A + (1+r)^2 A^2 + \dots) l, \quad ***$$

**vgl. jedoch Sraffa (11/), Appendix D3. Zur Identifizierung des korrekten Wertes von R bei mehreren möglichen s.h. weiter unten § 7 und Sraffa Appendix B und C. Zur Bestimmung der maximalen Profitrate darf nur das "Basissystem" (s.h. unten) in Betracht gezogen werden.

* s.h.u. § 17.

*** Beweis s.h. unten § 6.

die sich unmittelbar ökonomisch interpretieren lässt: die $A^k l$ stellen ja, wie wir gesehen haben, die vor k Jahren aufgewandte Arbeit, die indirekt in die gegenwärtige Produktion eingeht dar:

$$A^{(k)} l = L^{(k)}$$

So haben wir

$$p = w \left(L^{(0)} + (1+r) L^{(1)} + (1+r)^2 L^{(2)} + \dots \right),$$

d.h., jedes p_i/w lässt sich darstellen als die Summe aller in früheren Perioden zur Produktion des Gutes i augewandten Arbeit, wobei nur jeder Summand mit einer Potenz von $1+r$ zu multiplizieren ist, die angibt, um wieviel Jahre die betreffende Arbeitsleistung zurückliegt.

Hieraus lassen sich eine Anzahl von Schlüssen ziehen.

1) $\frac{p}{w}$ hat die Dimension "Zeit pro Einheit des Gutes i "; es ist der Preis des i -ten Gutes pro Lohnrate oder "pro Stundenlohn". Also gibt $\frac{p}{w}$ die Zeit an, die ein Arbeiter beim gegebenen w bzw. r arbeiten muss, um den Gegenwert einer Einheit des i -ten Guts zu verdienen*. Aus

$$\frac{p}{w} = L_0 + (1+r) L^{(1)} + (1+r)^2 L^{(2)} + \dots$$

folgt, dass diese Zeit für jedes Gut wächst, wenn r wächst, bzw. w fällt. Anschaulich: bei sinkendem Lohn muss sich ein Arbeiter um den Erwerb jeder beliebigen Ware länger mühen.

*Oder umgekehrt: "The value of any commodity, therefore, to the person who posses it, ... , is equal to the quantity of labour which it enables him to purchase or command". (Adam Smith, The Wealth of Nations, Vol. I, Seite 34 (/36/)).

Vgl. auch Straffa § 49, ferner § 32.

2) Dies steht zu Ricardos erstaunlicher Beobachtung, wonach der Preis gewisser Waren bei einer Lohnsteigerung fallen kann, nicht im Widerspruch, denn Ricardo meint den absoluten Preis in einem Standard, der sich von der Lohnrate unterscheidet. Wie wir gesehen haben, fällt r , wenn w steigt. Die einzelnen Preise in $p = w(I - (1+n)A)^{-\frac{1}{n}}$ werden sich also bei Bewegungen von w , zu denen die entsprechenden von r stets entgegengerichtet sein werden, verschieden verhalten*

Es leuchtet ein, dass bei einer (kleinen) Zunahme von w und der entsprechenden (kleinen) Abnahme von r der Preis einer mit direkter Arbeit ohne Zuzug anderer Produktionsmittel hergestellten Ware steigen und der einer mit geringer direkter Arbeit, dafür umso grössem Aufwand an anderen Produktionsmitteln hergestellten zweiten Ware fallen wird. Man wäre geneigt, den ersten Herstellungsprozess als 'arbeitsintensiv', den zweiten als 'kapitalintensiv' zu bezeichnen.

Es kann aber sein, dass der Schein trügt, und die Preisbewegungen anders ausfallen. Wenn zum Beispiel im zweiten Prozess die Produktionsmittel ganz besonders 'arbeitsintensiv' Güter sind, so werden ihre Preise mit w steigen und es kann sein, dass ihr Steigen mit w das Fallen von r übertrifft, sodass die produzierte Ware paradoxe Weise im Preise steigt.

Der Charakter der Fluktuationen der absoluten Preise hängt zu

* vgl. hierzu Sraffa §§ 13-22.

zum Teil vom gewählten Standard ab. Daher die Bedeutung der Konstruktion eines geeigneten Standards. Die Möglichkeit paradoxer Bewegungen ist aber schon bei relativen Preisen, wie wir gleich sehen werden, gegeben.

3) Aus

$$p = w \left(L^{(0)} + (1+r) L^{(1)} + (1+r)^2 L^{(2)} + \dots \right)$$

Sieht man, dass die Preisbewegungen anschaulich besser nicht aus dem Verhältnis von direkter Arbeit zu andern aufgewandten Produktionsmitteln abgeleitet werden, sondern sich exakt aus der zeitlichen Verteilung aller, d.h. direkter und indirekter, im Produkt verkörperter Arbeit, erklären. Dies sei an einem Beispiel Sraffas (der seinerseits Vorbilder hat) deutlich gemacht. Eine Flasche Wein (α) verkörpere zwanzig, acht Jahre zurückliegende Arbeitsstunden, eine Eichentruhe (β) ebensoviel, wovon jedoch eine fünfundzwanzig Jahre zuvor aufgewandt wurde, während die neunzehn verbleibenden in der Gegenwart geleistet wurden. Also:

$$p_\alpha = w (1+r)^8 \cdot 20,$$

$$p_\beta = w (19 + (1+r)^{25}).$$

Um w zu eliminieren, betrachten wir das Verhältnis $\frac{p_\alpha}{p_\beta}$, gehen also zu relativen Preisen über:

$$\frac{p_\alpha}{p_\beta} = \frac{20 (1+r)^8}{19 + (1+r)^{25}}.$$

Für $r=0$ sind Wein und Eichentruhe gleich viel wert ($p_\alpha = p_\beta$). Mit wachsendem r wird der Wein zunächst teurer als die Truhe, fällt dann jedoch wieder und für $r=17\%$ ist erneut $p_\alpha = p_\beta$.

Sraffa schreibt hierzu (11, §48):

"The reduction to dated labour terms has some bearing on the attempts that have been made to find in the 'period of production' an independent measure of the quantity of capital which could be used, without arguing in a circle, for the determination of prices and of the shares in distribution. But the case just considered seems conclusive in showing the impossibility of aggregating the 'periods' belonging to the several quantities of labour into a single magnitude which could be regarded as representing the quantity of capital. The reversals in the direction of the movement of relative prices, in the face of unchanged methods of production, cannot be reconciled with any notion of capital as a measurable quantity independent of distribution and prices."

7) Es wird dem aufmerksamen Leser nicht entgangen sein, dass im Beispiel von "Wein und Eichentruhe" die maximale Profitrate R nicht in Erscheinung getreten ist. Dies muss man darauf zurückführen, dass Wein und Eichentruhe hier zur Herstellung keine anderen Produktionsmittel als Arbeit erfordern. Wir betrachten aber eigentlich Systeme, bei denen einige der produzierten Güter auch als Produktionsmittel verwendung finden. Dann wird für das System als Ganzes stets eine endliche maximale Profitrate existieren**

** Vorausgesetzt, wir haben Einzelproduktindustrien, d.h. keine Kuppelproduktion vor uns. Bei Kuppelproduktion ist die Existenz von R eine schwierige Frage, die im Hauptteil dieses Buchs ausführlich abgehandelt wird.

Alle Güter können danach unterschieden werden, ob sie direkt und indirekt für alle andern Güter als Produktionsmittel Verwendung finden (sogenannte "Basisgüter") oder nicht ("Nichtbasisgüter"). Nichtbasisgüter können den Charakter reiner Konsumgüter haben (z.B. "Speiseeis") oder in die Produktion einiger anderer Nichtbasisgüter eingehen (z.B. "Eier"). "Wein" und "Eichentruhe" oben wären als Beispiele von Nichtbasisgütern in einem Gesamtsystem mit Basisgütern anzusehen. Nichtbasisgüter gehen nur dadurch in den ökonomischen Kreislauf ein, dass sie direkt oder indirekt dem Konsum aus Löhnen und Profiten dienen. Basisgüter wie "Kohle" und "Stahl" sind dagegen in jedem Fall direkt in den Kreislauf der Produktion einbezogen. Entsprechend lässt sich die Inputmatrix A und die Outputmatrix I nach geeigneter simultaner Umnummerierung von Industrien und Gütern aufteilen in Untermatrizen

$$A = \begin{bmatrix} A_1^1 & A_1^2 \\ A_2^1 & A_2^2 \end{bmatrix} = \begin{bmatrix} A_1^1 & 0 \\ A_2^1 & A_2^2 \end{bmatrix}, \quad I = \begin{bmatrix} I_1^1 & 0 \\ 0 & I_2^1 \end{bmatrix},$$

wobei A_1^1, A_2^2 quadratische Matrizen und I_1^1, I_2^1 Einheitsmatrizen gleicher Ordnung und wobei $A_1^2 = 0$. Dieser technischen Zerlegung des Systems entspricht eine ökonomische:

$$(1+r) A p + w l = p$$

zerfällt in

$$(1+r) A_1^1 p_1 + w l_1 = p_1$$

$$(1+r) (A_2^1 p_1 + A_2^2 p_2) + w l_2 = p_2,$$

wobei p_1 den Preisvektor der Basisgüter, l_1 den Arbeitsvektor der

Basisindustrien, p_2 den Preisvektor der Nichtbasisgüter, ℓ_2 den Arbeitsvektor der Nichtbasisindustrien bedeuten. Nun zeigt sich, dass bei gegebener Verteilung w, r und irgendwie gegebenem Preisstandard die Preise p_1 unabhängig von p_2 bestimmt sind,

$$p_1 = w (I_1^1 - (1+r) A_1^1)^{-1} \ell_1,$$

Während p_2 sich erst nach vorgängiger Berechnung von p_1 angeben lässt.

Damit die Volkswirtschaft nicht in zwei nicht-überlappende Teile man auseinanderfällt, hat die Existenz mindestens eines Basisgutes anzunehmen. Im System von Ricardos Korn-Korn-Modell ist Korn das einzige Basisgut.

Die hier betrachteten "natürlichen Preise" scheinen im Gegensatz zu "Marktpreisen" und im Gegensatz zu den Vorstellungen der neoklassischen Theorie unabhängig von der Nachfrage* zu sein. Nachfragekurven der Wirtschaftssubjekte waren zur Preisbestimmung nicht erforderlich. Der Schein trügt jedoch insofern, als der Geschlossenheit des Systems wegen der Gebrauch eines Guts als Produktionsmittel seinen Preis ebenso beeinflusst wie die Eigenschaften des Guts, eine Ware mit bestimmten Produktionskosten zu sein. Nur von den Preisen der Konsumgüter, niemals der Basisgüter, kann man sagen, dass sie rein von der Kostenseite determiniert sind. Das objektive Kriterium der Verwendung eines Basisguts in bestimmter Weise im gegebenen statischen System trägt zur Preisbestimmung genauso bei wie (nach Regelung der Verteilung) die Kosten seiner Inputs. Subjektive Präferenzen der Wirtschaftssubjekte dagegen fallen aus dem Spiel, sobald die Produktionsstruktur mit der Zusammensetzung des Volkseinkommens fest vorliegt.

*vgl. Sraffa § 7.

8) Mit dem Basissystem haben wir den letzten Begriff eingeführt, der uns noch fehlte, um Ricardos Frage nach dem geeigneten Standard für den absoluten Wert anzugehen. Wir suchen dazu eine Ware, die mit dem Korn im Korn-Kornmodell dies gemeinsam hat, dass das Verhältnis ihres Werts zu den in sie eingehenden Produktionsmitteln (ohne Arbeit) unabhängig ist von der Verteilung und dies gleichgültig, welcher Standard für die Preise gewählt wird. Eine solche Ware, wenn sie sich finden lässt, wird dann selbst einen geeigneten Preisstandard abgeben, da sie Veränderungen der Verteilung in einer sehr einfachen Weise transparent macht: wenn beide, Warenpreis (die Ware ist der Standard) sowie das Verhältnis vom Warenpreis zum Wert der Produktionsmittel (ohne Arbeit) konstant bleiben, so ist klar, dass jede Zunahme der Lohnrate (also des Lohns) in einer proportionalen Abnahme der Profitrate resultieren muss und umgekehrt.

Wir haben schon gesehen, dass die Preise, wie immer ihr Standard gewählt wird, in Folge von Verteilungsänderungen in manifacher Weise fluktuieren und dass der Charakter der Veränderung von der Zusammensetzung der Inputs, also vom Verhältnis von direkter Arbeit zu den andern Produktionsmitteln abhängt. Entsprechende Bewegungen treten auf in der Proportion von Warenpreis zum Wert der in sie verarbeiteten Produktionsmittel, und diese Proportion ist auch von Industrie zu Industrie verschieden. Mit einer Ausnahme allerdings: wenn der Lohn Null ist (bei der maximalen Profitrate), stehen die Werte von Erzeugnis und vorgeschoßinem Kapital bei allen Industrien im selben Verhältnis:

$$(1+R) A_p = P.$$

Dieses Verhältnis wird auch die Standardware charakterisieren.

Sofern das Basissystem aus mehr als einer Industrie besteht, ist nicht anzunehmen, dass eine einzelne Ware wird als Standardware dienen können, da alle anderen Basisgüter direkt oder indirekt in ihre/Produktion eingehen, und da deren Preise Fluktuationen unterliegen, die sich kaum, wie das erforderlich wäre, gerade kompensieren können. Wir nehmen also ein Warenbündel, bestehend aus den Basisgütern q_1, \dots, q_m (andere als Basisgüter brauchen offenbar nicht in Betracht gezogen zu werden). Wir schreiben $q = (q_1, \dots, q_m)$ als Zeilenvektor $q = (q_1, \dots, q_m)$ und wählen q so, dass

$$(I + R) q A_1^* = q,$$

wobei A_1^* das Basissystem bedeutet. (Diese "Eigenwertgleichung" lässt sich nach Frobenius eindeutig lösen).

In Worten: wir nehmen die Industrien des Basissystems nicht in den Proportionen, in denen sie ursprünglich gegeben sind, sondern multiplizieren jede (vergrössern oder verkleinern) mit einem solchen Faktor, dass das Aggregat aller Inputs, . h.

$$q A_1^* = q_1(a_1^*, \dots, a_m^*) + \dots + q_m(a_1^*, \dots, a_m^*)$$

dem totalen Produkt proportional ist (d.h. $q A_1^* = \frac{1}{1+R} q$).

q ist bestimmt nur bis auf einen Linearfaktor. Wir legen ihn fest, indem wir $q \cdot l = 1$ setzen.

$q(I_1^* - A_1^*)$, das Nettoprodukt der Volkswirtschaft in den veränderten Proportionen, definieren wir jetzt als Standardware, d.h. wir legen fest: $q(I_1^* - A_1^*) p = 1$ bei jeder Verteilung (für alle r bzw. w). In der Tat erfüllt $q(I_1^* - A_1^*)$ die Forderung, dass das Verhältnis vom Wert der produzierten Ware zu Wert der Produktionsmittel für alle

Verteilungen bei jedem Standard konstant bleibt, denn

$$q(I_1^* - A_1^*) \bar{p} = R q A_1^* \bar{p}$$

für alle $\bar{p} = \begin{pmatrix} \bar{p}_1 \\ \vdots \\ \bar{p}_m \end{pmatrix}$.

Aus

$$(I_1^* - A_1^*) p_1 = r A_1^* p_1 + w l_1$$

folgt, wenn $q(I_1^* - A_1^*) p_1 = 1$,

$$1 = q(I_1^* - A_1^*) p_1 = r q A_1^* + w q l_1 = \frac{r}{R} q (I_1^* - A_1^*) p_1 + w,$$

$$\boxed{1 = \frac{r}{R} + w};$$

es besteht also ein linearer Zusammenhang zwischen Lohn w und Profitrate r in diesem Standard.

Wenn der Wert der Standardware $q(I_1^* - A_1^*)$ auf die in $q(I_1^* - A_1^*)$ verkörperte Arbeit L_q zurückgeführt wird, so löst er sich auf in eine besonders einfache Reihe:

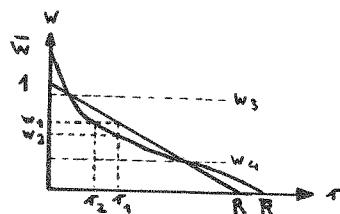
$$\begin{aligned} q(I_1^* - A_1^*) p_1 &= q(I_1^* - A_1^*) w (I_1^* - (1+r) A_1^*)^{-1} l_1 \\ &= R q A_1^* \left(1 - \frac{r}{R}\right) (I + (1+r) A_1^* + (1+r)^2 (A_1^*)^2 + \dots) l_1 \\ &= \left(1 - \frac{r}{R}\right) \frac{R}{1+R} q (I + (1+r) A_1^* + \dots) l_1 \\ &= \frac{R-r}{1+R} \left(q l_1 + \frac{1+r}{1+R} q l_1 + \left(\frac{1+r}{1+R}\right)^2 q l_1 + \dots\right) = 1. \end{aligned}$$

Straffa zeigt (§ 13-22), dass die Standardware mit Ricardos Konzept eines absoluten Wertmasses auch insofern übereinstimmt, als das Verhältnis von Arbeit und Kapital bei ihr dem Durchschnitt der Ökonomie entspricht.

- 9) Wenn in einem System, von dem wir der Einfachheit halber annehmen, es sei mit seinem Basissystem identisch, für eine bestimmte Ware eine alternative Produktionstechnik zur Verfügung steht, die der Reproduktionsbedingung $\sum_{i=1}^n \alpha_i^j \leq 1, j=1, \dots, n$ genügt, so erhebt sich die

Frage, mit welcher der beiden Techniken es sich ökonomischer produzieren lässt.

Die Frage ist leicht beantwortet: die Standardware des ursprünglichen Systems werde benutzt, um auch die Preise des Systems unter Benutzung der alternativen Technik zu normieren. Wir erhalten dann zwei $w-r$ -Diagramme. Die des ursprünglichen Systems ist eine Gerade durch $(r, w) = (0, 1)$ und $(r, w) = (R, 0)$. Die alternative Kurve wird davon verschiedene maximale Lohn- und Profitraten \bar{w} , \bar{R} aufweisen, die Gerade im allgemeinen bis zu n mal schneiden und in jedem Fall monoton fallen.



Für jede gegebene Lohnrate w_1 gehört die weiter rechts liegende Kurve zum jeweils ökonomischeren System, da dieses für diese Lohnrate eine höhere Profitrate $r_1 > r_2$ zur Folge hat. In Umgebung von r_1 wird umgekehrt dasselbe System bei gegebener Profitrate r_1 eine höhere Lohnrate $w_1 > w_2$ (d.h. einen höheren Lohn) erlauben. An Schnittpunkten w_3, w_4 sind beide gleichermaßen vorteilhaft.

Die Beobachtung, dass die $w-r$ -Kurven alternativer Techniken sich im allgemeinen mehrfach schneiden, ist ein schlagendes Argument gegen jede Theorie der Einkommensverteilung, die sich auf neoklassische Produktionsfaktoren stützt. Die Theorie der Produktionsfunktionen

steht und fällt mit der Existenz einer monotonen funktionalen Beziehung zwischen Kapitalintensität und Profitrate. Das Vorkommen mehrerer Schnittpunkte ist aber Ausdruck einer Wiederkehr derselben Technik bei verschiedenen Werten von r und widerlegt die Hypothese einer umkehrbar eindeutigen Zuordnung von Techniken und Profitraten*.

Andere Theorien der Einkommensverteilung wie die neokeynesianische Kaldors (/23/) stehen zum Phänomen der Wiederkehr der Techniken freilich nicht in Widerspruch. Dies darf wohl behauptet werden, obwohl eine umfassende, konsistente Darstellung, die dies unter Einbezug der Kapitalakkumulation und des technischen Fortschritts zeigen würde, noch aussteht. Joan Robinsons "The Accumulation of Capital" (/32/) ist wohl noch immer das Werk, das der Erfüllung dieser Aufgabe am nächsten kommt.

10) Da unser eigentliches Thema die in der neuen Literatur stiefmütterlich und in der Literatur über Sraffa (mit der im Vorwort erwähnten Ausnahme) überhaupt nicht behandelten "Kuppelproduktion" ist, werden wir nur im Nachwort noch einmal kurz auf das Verhältnis Sraffas zu anderen ökonomischen Doktrinen, insbesondere der Walras'schen Gleichgewichtstheorie, eingehen.

*Die Literatur hierzu ist schon fast unabsehbar: s.h. Harcourts Uebersicht (/21/).

Am Ende des Nachworts findet sich auch eine kurze Zusammenfassung
des nun folgenden Hauptteiles der Dissertation.

PART ONE: GENERAL JOINT PRODUCTION SYSTEMS

I Basics and Non basics

1. The System

Mr. Sraffa assumes that great numbers of production processes are available for the production of n given commodities, that a rate of profit is given, and that n of these processes (which together form a selfreproducing system) have been chosen so that the "prices of production" (see Sraffa, § 7) * at this one given rate of profit exist, are unique and positive. He then considers hypothetical variations in the rate of profit leading to hypothetical changes in the price system (changes in "demand" which might ensue, if the movement were to be real, are irrelevant in this context).

The equations are written as follows:

$$B_p = (1 + r) A p + w l \quad (1.1)$$

where $B = (b_i^j)$, $A = (a_i^j)$ are two non-negative (n, n) - matrices for outputs and inputs respectively. The lower index i refers to the processes represented by the rows a_i , b_i of A , B ; while the columns a^j , b^j refer to the commodities. We assume $a_i \neq 0$, $b_i \neq 0$, $i = 1, \dots, n$. In general, a vector with a lower index is a row-vector and with an upper index a column vector. All other vectors are column vectors. The transpose of a vector x is denoted by x^t . We have accordingly:

$$A = [a^1, \dots, a^n] = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = (a_i^j).$$

$e = \underbrace{[1, \dots, 1]}_n^t$ is the sum of all unit vectors $e_i = (0, \dots, 0, 1, 0, \dots, 0)$

* "Sraffa, § X" refers to "Production of Commodities by Means of Commodities", § X /11/.

and is used for the summing-up of the columns of a matrix:

$$e^t A = (e^t a^1, \dots, e^t a^n) = (\sum_{i=1}^n a_i^j).$$

$$e = \underbrace{(1, \dots, 1)}_n^t \text{ is an } n\text{-vector, } e_{(m)} = \underbrace{(1, \dots, 1)}_m^t$$

the analogous m -vector. I is the (n, n) -unit matrix, $I_{(m)}$ the (m, m) -unit matrix.

$\ell = (\ell_1, \dots, \ell_n)^t$ denotes the vector of the labour inputs, r , the rate of profit, w the wage-rate. As a convention, we fix $e^t \ell = 1$ (total labour-time expended equals one) and we measure each good in terms of its total output: $e^t B = e^t$.^{*}

We shall assume at first (see § 13) that $B - A$ is nonsingular:

$$\det(B - A) \neq 0. \quad (\text{A } 1)$$

It follows that the n processes $[a_i, b_i]$ are linearly independent:

$$\begin{aligned} \text{rk } [A, B] &= \text{rk } [A, B-A] = n \\ (\text{rk} = \text{rank}; [A, B] &= [a^1, \dots, a^n, b^1, \dots, b^n] \text{ is an } (n, 2n)\text{-matrix).} \end{aligned}$$

A further important general hypothesis we make is

$$e^t (B - A) = s^t \geq 0 \quad \text{**} \quad (\text{A } 2)$$

that is to say s , the vector of net products, shall be semi-positive, i.e. A, B is a selfreproducing system.

Starting from a rate of profit $r = r_0$ for which positive prices are supposed to exist we can now write our equation (1.1) as

$$w (B - (1+r)A)^{-1} \ell = p \quad (\text{1.2})$$

This is possible in an open neighbourhood of r_0 where

$$\det(B - (1+r)A) \neq 0.$$

Equation (1.2) determines the relative prices uniquely for given r .

* $e^t B = e^t$ does not exclude unproduced goods (see § 13).

** In matrix (vector) inequalities, the sign $A \leq B$ means $a_i^j \leq b_i^j$ and the sign $A \ll B$ means $A \leq B$ and $a_i^j < b_i^j$ for some (i, j) .

They may be normalized in various ways, e.g. either simply by putting $w = 1$ which amounts to expressing prices in terms of the wage-rate or by equalising a weighted sum of prices to a given constant, say

1. We define prices in terms of the wage rate as

$$\hat{p}(r) = (B - (1+r)A)^{-1} \ell$$

and we define prices in terms of a given commodity standard a (where $a = (a_1, \dots, a_n)$ denotes a vector of commodities) as

$$p_a(r) = \frac{(B - (1+r)A)^{-1} \ell}{a^T (B - (1+r)A)^{-1} \ell}$$

Thus, the wage rate in the standard a is

$$w_a = \frac{1}{a^T (B - (1+r)A)^{-1} \ell}$$

and we have

$$\hat{p}(r_o) > 0, \quad p_a(r_o) > 0$$

and $a^T p_a(r) = 1$ in a neighbourhood of r_o . The "basket of goods", a , may e.g. consist of only one good, say i , and then may be represented by a unit vector e_i .

$a = e$ has some mathematical advantages. Mr. Sraffa's standard prices are obtained, if a equals the standard (net) product. w_a , the wage rate in standard a , is also an indicator for the real wage, but only if the real wage is paid in the same standard a in which prices are measured. For if the real wage is $\lambda \cdot a$ (a is the "basket of goods", λ a scalar factor \approx "number" of "baskets"), we get, for the total wage, W ,

$$W = (e^T \ell) \cdot w_a = w_a = (e^T \ell) (\lambda a^T) \cdot p_a = \lambda.$$

An actual payment of the real wage in standard a presupposes of course

$$\lambda \cdot a^T \leq s^T = e^T (B - A),$$

while the measurement of prices in this standard does not require any such assumption.

Decomposability

If we want to be guided in our analysis by what we have learned in the case of single product industries, we have first to extend the distinction between basic and non-basic products.

By formal analogy we define:

Definition

The matrices A, B form an indecomposable system, if no permutation of rows and columns transforms A and B simultaneously into almost triangular matrices.

That is to say : A, B are indecomposable, if and only if no permutationmatrices* P, Q exist, so that

$$\bar{A} = PAQ = \begin{bmatrix} \bar{A}_1^1 & 0 \\ \bar{A}_1^2 & \bar{A}_2^2 \end{bmatrix} \quad \bar{B} = PBQ = \begin{bmatrix} \bar{B}_1^1 & 0 \\ \bar{B}_1^2 & \bar{B}_2^2 \end{bmatrix}$$

where \bar{A}_1^1, \bar{B}_1^1 are square matrices of the same order.

In the single product case one assumes without loss of generality that $Q = P^t = P^{-1}$ (so that single product systems (A, I) are indecomposable, if and only if A is an indecomposable matrix).

Decomposable systems are decomposable in the technical sense:
Let A, B be decomposable. A, B, B - A, and

$$(B-A)^{-1} = \begin{bmatrix} B_1^1 - A_1^1 & 0 \\ B_2^1 - A_2^1 & B_2^2 - A_2^2 \end{bmatrix}^{-1} = \begin{bmatrix} (B_1^1 - A_1^1)^{-1} & 0 \\ -(B_2^1 - A_2^1)^T(B_1^1 - A_1^1)^{-1}(B_2^2 - A_2^2)^{-1} & \end{bmatrix}$$

are then together almost triangular. A_1^1, B_1^1 form a group of, say k_1 processes which appears to be independent of the rest of the economy. (A_1^1, B_1^1) is in fact a self reproducing system, if $e_{(k)}^t (B_1^1 - A_1^1) \geq 0$. ($e_{(k)}^t (B_1^1 - A_1^1) \geq 0$ follows from $e^t (B-A) \geq 0$, if $B = I$, i.e. if (A, B) is a single product system.)

* Let $\bar{\pi} = (k_1, \dots, k_n)$ be a permutation of $(1, \dots, n)$.

Permutation matrices are

$$P = (p_i^j), p_i^j = \delta_{k_i}^j; \quad Q = (q_i^j), q_i^j = \delta_i^{k_j}; \quad \delta_i^j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Note: 1. PA is the matrix A with the rows, AQ with the columns permuted according to $\bar{\pi}$.

2. If P corresponds to $\bar{\pi}$, and if $\bar{\pi}^{-1}$ is the inverse permutation to $\bar{\pi}$, then P^{-1} corresponds to $\bar{\pi}^{-1}$.

3. $P^{-1} = P^t$ (permutation matrices are orthogonal).

Decomposability is illustrated best, if we assume constant returns to scale.

Interpreting q as activity levels in $e^t(B - A) = s^t \geq 0$, we may then change the activity levels to any $q \geq 0$ and denote the surplus by c :

$$c^t = q^t (B - A).$$

Considering $c \geq 0$ as given, c is said to be producible, if $c^t (B - A)^{-1} \geq 0$, for then

$$q^t = c^t (B - A)^{-1} \geq 0.$$

If $(B - A)^{-1} \geq 0$, (A, B) will form an all-productive system.^{*}

We shall call it all-engaging, if $(B - A)^{-1} \geq 0$ for then $q > 0$ for all $c \geq 0$. In an all-engaging system, all activities have to be engaged for the production of any good to take place.

It is well known that all single product systems $(A, B) = (A, I)$ are all-productive. A is indecomposable, if and only if $(B - A)^{-1} = (I - A)^{-1} \geq 0$, i.e. if and only if A, I is all-engaging.

Joint production systems are decomposable, if and only if an index $k < n$ exists so that (after permutations) for all $q = (q_1, \dots, q_k, 0, \dots, 0)^t$

it follows

$$c^t = q^t (B - A) = (c_1, \dots, c_k, 0, \dots, 0)^t.$$

(A, B) is all-productive in a decomposable system only if (A_1^1, B_1^1) is all-productive.

2. The Concept of a Basic System

The definition of indecomposability, though useful for the description of technical interdependence, is not satisfactory in joint production systems from the point of view of prices.

Consider the following example:

* It is easily seen (cf. § 8 below) that joint production systems are only very rarely all-productive. The concept is nevertheless important from the analytical point of view as we hope to show later.

An indecomposable system of n goods and processes $(A, B, \{ \})$ has an $(n + 1)$ st process added to it, producing a pure consumption good, say, "coke". The n -th process of the indecomposable system produces "gas". With the addition of the $(n + 1)$ st process, coke, which is always a technical by-product of gas, but which was regarded as a waste in the indecomposable system, becomes a joint product with gas. The n -th and the $(n + 1)$ st process differ only in that the n -th process employs a technique aimed at efficient gas-production, while in the $(n + 1)$ st process the emphasis is on coke.

The question now arises, whether coke, according to our assumption a pure consumption good, is "basic" or "non-basic". Since the system of $n + 1$ processes is also indecomposable, we have to find a new approach in order to show that here coke has a character akin to non-basics in single product industries, while gas plays the part of a basic. To this end, Mr. Sraffa proposes, (54), to deduct a suitable fraction of the $(n + 1)$ st process from the n -th so that coke production is cancelled out in a combined process with a positive output of gas but possibly some negative inputs. More generally, we define:

Definition

A system (A, B) is called non-basic, if a permutation of the columns and a number m exist so that the matrix $\begin{bmatrix} A^2 \\ B^2 \end{bmatrix}$ consisting of the last m ($1 \leq m \leq n - 1$) columns of A and of B has at most rank m .

Thus, coke as a pure consumption good ($a^{n+1} = 0$) is non-basic. If a system is non-basic in accordance with this definition, combined processes can be formed to eliminate the non basics:

If (A, B) is non basic, $n - m$ rows of the $(n, 2m)$ matrix $\begin{bmatrix} A^2 \\ B^2 \end{bmatrix}$ must be linearly dependent on at most m others. If they are the first, we represent them as a linear combination of the last m ($\begin{bmatrix} A^2_2 \\ B^2_2 \end{bmatrix}$) which are taken as a basis:

$$\begin{bmatrix} A^2_1 \\ B^2_1 \end{bmatrix} = H \begin{bmatrix} A^2_2 \\ B^2_2 \end{bmatrix}$$

and we construct a matrix (following Manara): $M = \begin{bmatrix} I_{(n-m)} & H \\ 0 & I_m \end{bmatrix}$
which transforms (A, B) into a pair of almost triangular matrices:

$$MA = \begin{bmatrix} A'_1 - HA'_2 & 0 \\ A'_1 & A'_2 \end{bmatrix}, \quad MB = \begin{bmatrix} B'_1 - HB'_2 & 0 \\ B'_2 & B'_2 \end{bmatrix}.$$

Any smallest such system (with the largest m) $\{[A'_1 - HA'_2], [B'_1 - HB'_2]\}$ will be called basic system. If it is identical with (A, B) , (A, B) will be called basic. The usual assumption in single product systems, that there should always be at least one basic good, will be replaced by the assumption that whenever we have (after permutation) $rk[A^1, B^1] = m$ for the last m columns of a system (A, B) or for any subsystem decomposed from a larger system (C, D) , it shall follow that

$$rk[A^1, B^1] > n-m. \quad (\text{A } 3)$$

If (A3) is not fulfilled, (A, B) can be made to fall apart into completely disconnected parts ('complete decomposability'). For if $rk[A^1, B^1] \leq n-m$, we get from $0 \neq \det(\bar{I}-A) = \det(M(\bar{I}-A)) = \det([(0, B'_2), (A'_1 - HA'_2, B'_2 - A'_2)])$

$$rk[A'_1 - HA'_2, B'_1 - HB'_2] = rk[A'_1 - HA'_2, (B'_1 - HB'_2) - (A'_1 - HA'_2)] = n-m.$$

Thus there is J with $[A'_1, B'_2] = [A'_1 - HA'_2, B'_1 - HB'_2]$ and $N = \begin{bmatrix} I_{(n-m)} & 0 \\ -J & I_m \end{bmatrix}$ such that $\det NM \neq 0$ and

$$NMA = \begin{bmatrix} A'_1 - HA'_2 & 0 \\ 0 & A'_2 \end{bmatrix}, \quad NMB = \begin{bmatrix} B'_1 - HB'_2 & 0 \\ 0 & B'_2 \end{bmatrix}.$$

Lemma: 1. If A, B basic, $A \geq 0, B \geq 0$, then $e^t A \geq 0, e^t B \geq 0$. **
2. If $e^t p = \lambda A p, p \geq 0$, then $\lambda \geq 1$.

Theorem 2.1

1. Single product systems are basic, if and only if they are indecomposable.
2. Assume $\det(\bar{I}-A) \neq 0, \det(\bar{I}-B) \neq 0$. The systems $(\bar{B}^{-1}A, I), ((\bar{B}-A)^{-1}A, I), ((\bar{A}-B)^{-1}B, I)$ etc. are indecomposable, if and only if (A, B) is basic.
- 2'. $(\bar{B}^{-1}A, I)$ etc. are completely decomposable, if and only if (A3) does not hold.
3. All-engaging systems are basic.

Proof: (Lemma) 1. Trivial. 2. Normalize p , so that $e^t p = 1$. With $e^t(\bar{B}-A) = e^t - e^t A \geq 0$ we get $1 = e^t p = e^t B p = \lambda e^t A p < \lambda e^t p = \lambda$.

Proof: (Theorem) 1. If (A, I) nonbasic, we get (after perm.) for the

* For the problem of uniqueness see §3.

** Not true, if only indecomposability is assumed.

last m columns $\text{rk}[A^2, I^2] \leq m$. From $\text{rk} I_2^2 = m$ follows

$$\text{rk}[A_1^2, I_2^2] = m, \text{ hence } A_1^2 = 0, \text{ since } I_1^2 = 0,$$

$A_1^2 \neq 0$. Thus A is decomposable. The converse is obvious.

2. If (A, B) is non basic, permutation matrices P, Q and a non-singular matrix M exist, so that $MPAQ$ and $MPBQ$ are almost triangular. Thus, the matrix $(MPBQ)^{-1} MPAQ = Q^{-1} B^{-1} P^{-1} M^{-1} MPAQ = Q^{-1} B^{-1} AQ$ is almost triangular as the product of two almost triangular matrices and so is $Q^{-1} IQ = I$.

Conversely, if $(B^{-1} A, I)$ is decomposable, (A, B) is non basic; for $Q^{-1} B^{-1} AQ$ is almost triangular (the last m columns are zero in the first $n - m$ rows), thus

$$m = \text{rk}[Q^{-1} B^{-1} AQ^2, I^2] = \text{rk}((B \cdot Q)[Q^{-1} B^{-1} AQ^2, I^2])$$

$$= \text{rk}[AQ^2, BQI^2] = \text{rk}[AQ^2, BQ^2],$$

thus m columns of A and B form together an $(n, 2m)$ -matrix of rank m . Similarly for $(B-A)^{-1} A$, $(B-A)^{-1} B$, and the proof of 2.1.2'.

3. Since $e^T B = e^T B \geq 0$, we have $(B-A)^T B > 0$, if $(B-A)^T > 0$.

Thus $(B-A)^T B$ is an indecomposable matrix, $((B-A)^T B, I)$ an indecomposable system, $((B-A)^T B, I)$ and (A, B) basic systems.

q.e.d.

Multiplication by M decomposes our original equation (1) into two separate expressions:

$$(1+r)(A_1^1 - HA_2^1)p_1 + w(I_1 - Hl_2) = (B_1^1 - HB_2^1)p_1 \quad (2.1)$$

$$(1+r)(A_2^1 p_1 + A_2^2 p_2) + wl_2 = B_2^1 p_1 + B_2^2 p_2.$$

The first of these equations can be solved independently of the second; both taken together give the same result as (1).

If $H = 0$, we are back to (technical) decomposability in the sense defined above. (A_1^1, B_1^1) then form a system by themselves, if $e_{(w)}^T (B_1^1 - A_1^1) \geq 0$. The technical independence of A_1^1, B_1^1 is here reflected in the fact that one need not know the

processes (A_2, B_2, ℓ_2) to determine prices in the system
(A_2^1, B_2^1, ℓ_2)

(A_2, B_2, ℓ_2) on the other hand should not be viewed as a block of interdependent processes or as a system opposed to (A_2^1, B_2^1, ℓ_2) but rather as a collection of disconnected industries producing a set of "luxury goods" none of which is technically essential or economically relevant for the basic part of the system.

Note that necessities of life of which the subsistence wage of the workers consists are basic goods par excellence. They include whatever is considered indispensable for the maintenance of the customary way of life of the labour force and they ought therefore to be implicit in the inputs of the whole economy and the outputs of the basic system.

This would involve splitting the wage into a basic part and a surplus part - a step which Mr. Sraffa is reluctant to take for the reasons expounded in his § 8. We shall leave the question open, since it is largely a matter of interpretation and does not affect the mathematical structure of the model. But we draw the reader's attention to the fact that a few of the propositions we are going to derive make sense only to the extent that the distinction between basic wage (to be advanced) and surplus wage (share in the surplus) is acceptable.

Let us return to the equations (2.1). If $H \neq 0$, it is not possible any more to say that a group of processes is technically independent of the others: the expressions $A_2^2 = HA_2^1$, $B_2^2 = HB_2^1$ show that some goods produced and/or used in the non-basic processes (the non-basics) are also used and/or produced in some basic processes. However, although the prices of the first $n-m$ goods are thus not independent of the last m processes (A_2, B_2, ℓ_2), we have at least constructed an imaginary system

$$(A_2^1, B_2^1, \bar{\ell}_2) = (A_2^1 - HA_2^1, B_2^1 - HB_2^1, \ell_2 - H\ell_2)$$

which determines the prices of the first $n-m$ goods for all rates of profit in such a way that they appear to be independent of the movements of the prices of the last m goods.

This procedure is justified by the demonstration (given in the next paragraph) that except for flukes no mathematical procedure exists with which it is possible to decompose the system further than into basics and non-basics.

3. The Uniqueness of the Basic System

The uniqueness of the basic system has to be proved in a double sense: firstly we have to show that only one basic system exists, if the Sraffian definition is accepted. Secondly, we have to solve the problem set out in the last paragraph; that is, we have to show, that no other definition of the basic system could be given (in a sense still to be specified).

We take the second problem first and begin by analysing any two matrices F, G with a labour-vector, m , satisfying the same price system as equation (1.1)

$$(G - (1+r)F) \hat{P}(r) = m \quad (3.1)$$

where

$$\hat{P}(r) = (B - (1+r)A)^{-1} \ell. \quad (3.2)$$

(F, G, m) shall fulfill the same assumptions as (A, B, ℓ) . In particular $\det(B-A) \neq 0$, $\det(G-F) \neq 0$.

Write A_{Ad} for the adjoint of a matrix A . If $\det A \neq 0$, $A_{Ad} = A^{-1} \det A$, if $\det A = 0$, $A_{Ad} = 0$. (3.1), (3.2) are equivalent to

$$det((B - (1+r)A) \cdot D\ell) = [G - F - rF] [B - A - rA]_{Ad} \ell \quad (3.3)$$

where we choose D to be any nonsingular matrix mapping ℓ onto m :

$$m = D\ell.$$

Expand the two sides of (3.3) to get $((B - (1+r)A)_{Ad}$ is a polynomial matrix of degree $n-1$ in r):

$$\sum_{v=0}^n \beta_v r^v D\ell = [(G - F - rF)(z^0 + z^1 r + \dots + z^{n-1} r^{n-1})] \quad (3.4)$$

with

$$\beta_v = \det(B - A)$$

$$z^0 = (B - A)_{Ad} \ell.$$

As (3.1), (3.2) shall hold for a full neighbourhood of r_0 , (3.4) will

hold identically in r and is equivalent to the equations

$$\begin{aligned}\beta_0 Dl &= (G - F) z^0 \\ \beta_1 Dl &= (G - F) z^1 - F z^0 \\ &\vdots \\ \beta_{n-1} Dl &= (G - F) z^{n-1} - F z^{n-2} \\ \beta_n Dl &= -F z^{n-1}\end{aligned}\quad (3.5)$$

which we write in matrix form:

$$[G - F, -F] \begin{bmatrix} z^0, z^1, \dots, z^{n-1}, 0 \\ 0, z^0, \dots, z^{n-2}, z^{n-1} \end{bmatrix} = [\beta_0 Dl, \dots, \beta_n Dl]. \quad (3.6)$$

We ask now, to what extent $[G - F, -F]$ is determined by (3.6). This will depend on the rank of the (n, n) -matrix $[z^0, \dots, z^{n-1}]$. First, it is clear that

$$[G - F, -F] = D[B - A, -A] = [DB - DA, -DA] \quad (3.7)$$

can be taken as a particular solution of (3.6), since D was chosen to be invertible. The complete set of solutions to (3.6) can be represented as the sum of this particular solution (3.7) plus all the solutions of the corresponding homogenous equation, written as:

$$[X, Y] \begin{bmatrix} z^0, \dots, z^{n-1}, 0 \\ 0, z^0, \dots, z^{n-1} \end{bmatrix} = 0. \quad (3.8)$$

Any X and Y for which

$$X[z^0, \dots, z^{n-1}] = 0, \quad Y[z^0, \dots, z^{n-1}] = 0$$

that is to say, for which $X \hat{\rho}(r) = 0, \quad Y \hat{\rho}(r) = 0$ identically in r , will satisfy this equation, but these are not all the solutions. We get a complete survey as follows:

Consider any X fulfilling

$$X z^0 = X(B - A)_{Ad} l = 0 \quad (3.9)$$

to be given. (3.8) will then determine Y :

$$X[z^1, \dots, z^{n-1}, 0] = Y[z^0, \dots, z^{n-1}] \quad (3.10)$$

but not fully, unless $\det[z^0, \dots, z^{n-1}] \neq 0$.

Retracing our path to the original equations, we find from

$$(X - rY) [(B-A) - rA]^{-1} \ell = 0$$

that

$$Y_0 = X(B-A)^{-1}A \quad (3.11)$$

is a particular solution to (3.8), because

$$\begin{aligned} [X - rY] [(B-A) - rA]^{-1} \ell &= X(I - r(B-A)^{-1}A) [(B-A) - rA]^{-1} \ell \\ &= X(B-A)^{-1} \ell = 0. \end{aligned}$$

To get the full set of solutions to (3.10), we have again to add to Y_0 the solutions Y_1 of the corresponding homogeneous equation

$$Y_1 [z^0, \dots, z^{n-1}] = 0.$$

These solutions are the same as the ones for which

$$Y_1 \hat{p}(r) \equiv 0 \quad (3.12)$$

identically in r .

Our matrices F, G with the labour vector m are now seen to be related to the original A, B, ℓ by

$$\begin{aligned} G - F &= D(B-A) + X \\ F &= DA + Y_0 + Y_1 \end{aligned} \quad (3.13)$$

where X is any matrix satisfying (3.9), Y_0 is related to X by (3.11), and Y_1 satisfies (3.12). Define

$$M = D + X(B-A)^{-1}$$

and (3.13) becomes

$$G - F = M(B-A)$$

$$F = MA + Y_1$$

and

$$M\ell = D\ell + X(B-A)^{-1}\ell = m,$$

where
 $\det M \neq 0$, for $\det(G - F) \neq 0$, $\det(B - A) \neq 0$. Since our derivation can be reversed we have found:

Theorem 3.1

Prices are the same in two systems (F, G, m) , (A, B, ℓ) for all rates of profit, i.e.

$$\hat{p}(r) = (G - (1+r)F)^{-1}m = (B - (1+r)A)^{-1}\ell,$$

if and only if matrices M, Y with $\det M \neq 0$, $Y \hat{p}(r) \equiv 0$ exist

so that

$$F = MA + Y, \quad G = MB + Y, \quad m = M\ell.$$

Corollary: Prices $\hat{p}(r)$ are the same in two systems (F, G, m) , (A, B, ℓ) for all rates of profit, if they coincide in $n+1$ different points r_0, \dots, r_n ($r_i \neq r_j$).

Proof (of Corollary): Immediate by application of the theorem of identity for polynomials (equation (3.4) holds in $n+1$ points).

The two matrices M and Y , by which (F, G, m) and (A, B, ℓ) are related to each other, have straightforward economic interpretations:

To premultiply A, B, ℓ by M means to form n new processes f_i, g_i, l_i by linear combination from the original a_i, b_i, l_i .

To add Y with $\hat{Y}(r) \equiv 0$ means to add artificially to each process a set of inputs and outputs whose value is zero at all rates of profit. Thus, in the first case, new industries are formed by linear superposition; in the second industries "exchange" goods in such quantities that the value of the exchange is zero at all rates of profit and in every industry.

To illustrate the second operation which one might think impossible take an indecomposable matrix as the input matrix of a single product system (A, I, ℓ) and assume the labour vector to be the positive eigenvector belonging to the dominant root of A . This is the famous special case for which the "organic composition of capital" is the same in all industries and independent of the rate of profit. If

$$(1+r) A p + w \ell = p$$

and

$$A \ell = \frac{1}{1+r} \ell,$$

it follows

$$p(r) \equiv \ell \text{ for } w = \frac{R-r}{1+r}.$$

Define

$$y^1 = -\frac{1}{R} (a^1 \ell_2 + \dots + a^n \ell_n), \quad y^2 = a^1, \dots, \quad y^n = a^n,$$

$$Y = [y^1, \dots, y^n].$$

In the system

$$(1+r)\bar{A}\hat{p} + \omega l = \hat{p}$$

with

$$\bar{A} = A - Y$$

which is equivalent with (A, I, ℓ) in that prices are the same as in (A, I, ℓ) , $n-l$ goods are "non-basics" despite the indecomposability of A .**

Next, we show that one is justified in neglecting matrices Y with $Y \hat{p}(r) \equiv 0$, for such matrices $Y \neq 0$ cannot exist unless A, B is a mathematically and economically exceptional system and and/or ℓ is in a very special relationship with A, B .

In slight modification of the conventional terminology, we call a root R of $\det(B - (1+r)A) = 0$ semi-simple, if $\text{rk } (B - (1+R)A) = n-l$. Whether R is a simple root or not: if R is semi-simple, there is (up to a scalar factor) one and only one, "eigenvector" q with $q(B - (1+R)A) = 0$.

Theorem 3.2

Let R_1, \dots, R_t be the roots of $\det(B - (1+r)A) = 0$, with multiplicities s_1, \dots, s_t , ($s_1 + \dots + s_t = n$). The price vector $\hat{p}(r)$ assumes n linearly independent values $\hat{p}(r_1), \dots, \hat{p}(r_n)$ at any n different rates of profit r_1, \dots, r_n ($r_i \neq r_j, r_i \neq R_j$), if all t roots R_1, \dots, R_t of the equation $\det(B - (1+r)A) = 0$ are semi-simple and if $q_i \ell \neq 0, i=1, \dots, t$, for the associated eigenvectors q_i . Conversely, if one root is not semi-simple or if $q_i \ell = 0$ for some i , it follows that $\hat{p}(r_1), \dots, \hat{p}(r_n)$ are linearly dependent for any r_1, \dots, r_n ($r_i \neq R_j$). If \bar{R} is real, there is a real vector \bar{q} with $\bar{q} \hat{p}(r) = 0$ for all r .

Proof Let s_i be the multiplicity of R_i , R_i semi-simple, $\sum_{i=1}^t s_i = n$. According to Jordan's theory of Normal Forms*, there exist n linearly

* see e.g. W. Gröbner, /3/, p.201-205.

** This result may be used to show that Samuelson's economy for which he constructed a 'surrogate production function' was not what we shall call a truly basic system and that it could have been reduced to a one-product economy. (He assumes implicitly that prices are constant, i.e. that his prices are equal to values. See /13/.)

independent vectors

$$q_{i,1}, \dots, q_{i,s_i}; \quad i = 1, \dots, t;$$

with

$$\begin{aligned} q_{i,1} &= q_{i,1} \\ q_i &= R_i q_i A (B - A)^{-1}, \\ q_{i,\sigma} &= (B - (A + R_i) A) (B - A)^{-1} \\ &= q_{i,\sigma} (I - R_i A (B - A)^{-1}) \\ &= -R_i q_{i,\sigma-1} \quad \sigma = 2, \dots, s_i. \end{aligned}$$

It follows

$$q_{i,\sigma} (I - \tau A (B - A)^{-1}) = q_{i,\sigma} \left(1 - \frac{\tau}{R_i}\right) - \tau q_{i,\sigma-1}$$

and this formula holds for $\sigma = 1, \dots, s_i$, if we define $q_{i,0} = 0$ for all i .

With this we get

$$\begin{aligned} q_{i,\sigma} (B - A) \hat{p}(r) &= q_{i,\sigma} (B - A) (B - (A + rA) A)^{-1} l \\ &= q_{i,\sigma} (I - \tau A (B - A)^{-1})^{-1} l \\ &= \frac{R_i}{R_i - \tau} q_{i,\sigma} l + \frac{\tau R_i}{R_i - \tau} q_{i,\sigma-1} (I - \tau A (B - A)^{-1})^{-1} l \\ &= \frac{R_i}{R_i - \tau} q_{i,\sigma} l + \frac{\tau R_i}{R_i - \tau} \frac{R_i}{R_i - \tau} q_{i,\sigma-1} l \\ &\quad + \frac{(R_i)^2}{(R_i - \tau)^2} q_{i,\sigma-2} (I - \tau A (B - A)^{-1})^{-1} l \\ &= \frac{R_i}{R_i - \tau} q_{i,\sigma} l + \frac{\tau R_i^2}{(R_i - \tau)^2} q_{i,\sigma-1} l \\ &\quad + \dots + \frac{R_i}{R_i - \tau} \left[\frac{\tau R_i}{R_i - \tau} \right]^{\sigma-1} q_{i,1} l; \\ &\quad \sigma = 2, \dots, s_i; \\ &\quad i = 1, \dots, t; \\ q_i (B - A) \hat{p}(r) &= \frac{R_i}{R_i - \tau} q_i l. \end{aligned}$$

Define

$$Q = [q_{1,1}^1, \dots, q_{1,s_1}^1, \dots, q_{t,1}^1, \dots, q_{t,s_t}^1],$$

$$T = Q(B - A).$$

The vector

$$v(r) = T \hat{p}(r)$$

assumes in n points r_1, \dots, r_n ($r_i \neq r_j, r_i \neq R_i$) n linearly independent values, if and only if $q_i \cdot l \neq 0; i = 1, \dots, n$. The necessity of this latter condition is obvious, for

$q_i(B-A)\hat{p}(r) = 0 \quad \text{if } q_i \cdot l = 0$. To verify that $q_i \cdot l \neq 0$ is sufficient, consider the matrix

$$N = [n(r_1), \dots, n(r_n)]$$

where $n(r) = [\det(B - (1+r)A)] v(r)$. Assume that N (consisting of the values of n polynomials at n points) is singular, i.e. assume, the s_1 -th row is equal to a linear combination of the first $s_1 - 1$ and the $n - s_1$ last rows. Since the values in n points determine a polynomial of $(n-1)$ st degree (and none of the polynomials is of higher degree) fully, the polynomial in the s_1 -th row would have to be equal to the linear combination of the first $s_1 - 1$ and the $n - s_1$ last rows not only in r_1, \dots, r_n but everywhere. But this is impossible, since the s_1 -th polynomial does not vanish at $r = R_1$ while the $n - 1$ other polynomials and hence their linear combination are zero at $r = R_1$. Thus N is non-singular and

$$\hat{p}(r) = T^{-1} v(r)$$

assumes n linearly independent values at n different points, if the R_i are semisimple, and if and only if $q_i \cdot l \neq 0, i = 1, \dots, t$.

The necessity of the R_i being semisimple remains to be shown.

Suppose R is a multiple root of $\det(B - (1+R)A) = 0$ and

$\operatorname{rk}(B - (1+R)A) < n-1$. There are then two linearly independent q_1, q_2 with

$$q_i(B - (1+R)A) = 0; i = 1, 2; q_i \cdot l \neq 0.$$

We get as above

$$q_i(B-A)(B-(1+R)A)^{-1}l = q_i(I - r A (B-A)^{-1})^{-1}l = \frac{R}{R-r} q_i \cdot l; i = 1, 2;$$

thus

$$(q_1 - \lambda q_2)(R - A)\hat{p}(r) \equiv 0, \lambda = \frac{q_1 l}{q_2 l},$$

identically in r with $(q_1 - \lambda q_2)(R - A) \neq 0$. This is impossible, if $\hat{p}(r)$ assumes n linearly independent values in any n points.

$\hat{q} = q_1 - \lambda q_2$ is a real vector, if R is real.

q.e.d.

Multiple roots which are not semi-simple are mathematically rather exceptional^{**} and the careful examination of examples suggests that they are hardly more than flukes in any system from the economic point of view. We exclude them therefore at present from our considerations but shall revert to them in §7.

As regards the condition that $q_i l \neq 0$ for all eigenvectors q_i (the roots being semisimple), there is because of $l > 0$, a part from a finite number of mathematical flukes,^{***} again no economically relevant case except the one we dealt with separately above where l is equal to the prices at the maximum rate of profit and the organic composition is constant: if l is an eigenvector $A l = (1+R)^A l$, it follows $q_i l = 0$ for all q_i belonging to different eigenvalues $R_i \neq R$.

Barring this, $\hat{p}(r)$ assumes n linearly independent values for any n different given rates of profit, whether A, B is basic or not. A number of results spring from this unexpected property of prices in

Even multiple roots are exceptional. for it follows from the theory of algebraic functions that every multiplicity of roots is unstable.

^{**}at most n .

^{***}This follows from

$$\frac{1}{1+R_i} q_i l = q_i A l = \frac{1}{1+R} q_i l, l_i \neq R.$$

^{****}Thus, prices change direction erratically (and yet continuously) in what we shall call regular systems. Moreover, one can show that nor absolute nor relative prices are constant for any interval, be it ever so small, in regular systems.

Sraffa-systems (we may call systems with this property regular and the exceptions irregular). It follows e.g., that relative prices $\frac{P/w}{P/lw} = \frac{P}{l}$ in a regular two sector model rise (or fall) monotonically with r . Or we have:

Corollary to Theorem 3.1 and 3.2

Two regular single product systems $(A_1, I, l^1), (A_2, I, l^2)$ whose price vectors coincide at $n + 1$ rates of profit are identical.

Proof: The coincidence of the prices at $n + 1$ points is sufficient to allow the application of theorem 3.1. Therefore $A_1 = MA_2 + Y$, $Y = 0$, because prices assume linearly independent values at any n of the $n + 1$ points. $M = I$, because the output matrices are both unit matrices. Thus $A_1 = A_2$.

q.e.d.

Let us return to the discussion of the definition of a basic system.

If (A, B, \emptyset) is regular, (F, G, m) is regular, no $Y \neq 0$ with $Yp(\emptyset) \neq 0$ can exist and $F = MA$, $G = MB$, and $m = Ml$, $\det M \neq 0$, for any two systems with $\hat{p} = (B - (1+r)A)^{-1}l = (G - (1+r)F)^{-1}m$. Assume now, $F = MA$, $G = MB$ are almost triangular, i.e. assume the last m columns are zero in the first $n - m$ rows. Since M is non-singular,

$$rk[MA^2, MB^2] = rk[A^2, B^2] \leq m.$$

Thus, A, B is non basic and the multiplication by M which turned A, B into almost triangular matrices, could also be effected by a matrix of the form $\begin{bmatrix} I_{(n-m)} & -H \\ 0 & I_m \end{bmatrix}$ as used in the definition of the basic system.

Note that any system decomposed in this way from a regular system is itself regular. Proof: The goods belonging to the decomposed system are uniquely determined and so are its prices $p_i(r)$ (using the usual notation). If the decomposed system of $n - m$ goods were not regular, a not necessarily real $(n - m)$ -vector $q_1 \neq 0$ would exist such that $q_1 p_i \neq 0$. But then $[q_1, 0] \begin{bmatrix} p_1(r) \\ p_2(r) \end{bmatrix}$ for the n -vector $[q_1, 0] \neq 0$ in contradiction to the assumption that the system is regular, i.e. that $p(r)$ assumes n linearly independent values for any n rates of profit.

Thus we have proved:

Theorem 3.3

Any system decomposed* from a regular system is itself regular and is (up to permutations and linear combinations of processes) identical to one which can be arrived at by means of the procedure used in the Sraffa-Manara definition of the basic system (§ 2).

The second of the two problems set at the beginning of this paragraph is now solved for regular systems: We have shown that by no other definition than the one given in § 2 we can decompose a regular system further than into a basic system and a group of non-basic processes. For if a second decomposition were possible, it could also be effected by the Sraffa-Manara construction, and the result of the first could not have been the basic system, since the basic system was defined as the smallest system arrived at this way (its uniqueness remains to be shown). Since the decomposed system of a regular system is regular, the same argument applies, if a decomposed system is decomposed once more in itself.

We were able to show the uniqueness of the definition of the basic system in the regular case, because we did not alter our concept of a system. For if variable transformation matrices are admitted and systems with variable coefficients introduced, do further decompositions become possible. To deal with fixed capital, we shall replace the matrix M by matrix polynomials in $l+r$, and an old machine will then be revealed as a sort of non-basic in equations similar to (2.1), but with variable coefficients which are rational functions in $l+r$.

As we can see from the example after the corollary of theorem 3.1, the assumption of regularity is also essential in this context: a non-regular basic system can sometimes be decomposed by means of the Y -matrices. (see also § 7). We therefore call a

* i.e. by means of M - and Y -matrices

regular basic system truly basic and an irregular basic system pseudo-basic.

Corollary: The basic system of a regular non-basic system is truly basic.

As regards irregular systems, the Sraffa-Manara definition has not been justified by our considerations (except in that irregular systems have mathematically and economically the character of exceptions). The decomposition of a basic system by means of Y-matrices as shown above (industries 'exchange' a set of goods of zero value) is economically as meaningful as the construction of the basic system (linear combination of industries). We shall return to pseudo-basic systems in §7.

We have finally to prove an assertion we have often made (although of course not used in proofs), namely that the Sraffa-Manara construction leads to a uniquely determined basic system both in the regular and the irregular case.

Theorem 3.4

Two basic systems derived from the same system (A, B, L) by means of the Sraffa-Manara construction (§2) consist of the same goods (with the same prices for all r) and are identical up to linear combinations of the processes.

Proof:

Following theorem 2.1 it is evidently sufficient to show that the basic part of $(B-A)^{-1}B, I$ is unique. Following Gantmacher (1, XIII, § 4) the system $((B-A)^{-1}B, I)$ can be decomposed after simultaneus permutations of rows and columns in the following manner:

$$D = (B - A)^{-1} B = \begin{bmatrix} D_1^1 & \dots & D_1^2 & \dots & \dots & \dots & D_1^n \\ D_2^1 & \dots & D_2^2 & \dots & \dots & \dots & D_2^n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ D_g^1 & \dots & D_g^2 & \dots & \dots & \dots & D_g^n \\ D_s^1 & \dots & D_s^2 & \dots & \dots & \dots & D_s^n \end{bmatrix}$$

with each D_1^1, \dots, D_s^s indecomposable and so that in each sequence D_f^1, \dots, D_f^{f-1} ($f = s+1, \dots, s$) at least one matrix is different from zero. The number g is then uniquely determined. With theorem 2.1 it follows from assumption (A3) applied to the completely decomposable system

$$\begin{bmatrix} D_1^1, 0, \dots, 0 \\ 0, D_2^2, \dots, 0 \\ \vdots & \ddots & \vdots \\ 0, \dots, 0, D_g^g \end{bmatrix}, \quad \begin{bmatrix} 1, 0, \dots, 0 \\ \vdots & \ddots & \vdots \\ 0, \dots, 0, 1 \end{bmatrix}$$

that $g=1$. Thus

$$D = \begin{bmatrix} D_1^1 & 0 \\ 0 & D_2^2 \end{bmatrix}$$

with D_1^1 indecomposable and uniquely determined. The rest is obvious.

q. e. d.

In §20 we shall give an example of a system of three goods and three processes which can be constructed by subtracting any of the three processes from the two others. That example will make it plain that the Sraffa-Manara construction can indeed lead to basic systems involving the same goods but identical only up to linear combinations of the processes.

II. Measurement and Distribution of the Surplus

4. Labour Values

In single product systems, $p_i(o) = u_i$ is interpreted as the amount of direct and indirect labour embodied in one unit of good i . The same can be said in the case of joint production: $a_{ij}u + l_i$ is the amount of direct and indirect labour embodied in process i . This labour is expended in different proportions on the products b_{ij} which constitute the output of process i :

$$a_{ij}u + l_i = b_{ij}u$$

or

$$Au + l = Bu,$$

hence

$$u = (B - A)^{-1}l.$$

In the single product systems u can be resolved into the sum of past labour inputs: Al is the labour being embodied in the current period, A^2l the labour embodied in the preceding period, and so on:

$$u = l + Al + A^2l + \dots$$

This formula is mathematically proved by matrix expansion

$$(I - A)^{-1} = I + A + A^2 + \dots$$

valid, as is well known, for all A whose eigenvalues are all smaller than one in absolute value and in particular for all matrices representing selfreproducing single product systems.

A similar expansion is not always possible if $B \neq I$, nor is there an economic reason to expect it (see § 19 on land). We have, however,

$$(I - B^{-1}A)^{-1} = I + B^{-1}A + \dots$$

if $\det(B - \lambda_i A) = 0$ entails $|\lambda_i| > 1$ for all n roots of the equation; for $\det(B - \lambda_i A) = 0$ entails $\det(\frac{1}{\lambda_i}I - B^{-1}A) = 0$.

The expansion of the labour vector

$$u = (I - B^{-1}A)^{-1}B^{-1}l = B^{-1}l + B^{-1}AB^{-1}l + \dots$$

is to date quantities of labour of the transformed system $(B^{-1}A, I, B^{-1}l)$.

A third way of accounting labour-values is the so-called subsystems approach (Sraffa, Appendix A):

Write S for the diagonal matrix derived from the surplus vector (assume the surplus to be strictly positive or use multipliers to make it strictly positive):

$$S = \begin{bmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{bmatrix}; \quad s^i = (s_1, \dots, s_n) = e^i(B-A) > 0.$$

We determine activity levels q_i by which the surplus s_i of good i and nothing else is produced:

$$q_i = s_i [e_i (B-A)^{-1}], \quad e_i = (0, \dots, 0, 1, 0, \dots, 0)$$

or

$$Q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = S (B-A)^{-1}.$$

$$e^i Q = e^i S (B-A)^{-1} = s^i (B-A)^{-1} = e^i$$

means that we may interpret Q as a virtual subdivision of the actual activity levels e . Since q_i are the activity levels appropriate for the production of s_i ,

$$\bar{u}_i = q_i \cdot l$$

must be the amount of work expended on the isolated production of s_i and $\tilde{u}_i = \frac{1}{s_i} u_i$ expresses the amount of direct and indirect labour embodied per unit of good i .

We have

$$\tilde{u} = S^{-1} \bar{u} = S^{-1} Q l = S^{-1} S (B-A)^{-1} l = (B-A)^{-1} l = u$$

as was to be expected.

An Apparent Paradox *

Suppose A , B , l is such that one of these prices $p(0)$ at $r = 0$ or "labour values", say $p_1(0) = u_1$, is negative. Consider an expansion of the production of good 1 by some small amount δ_1 . q is the vector of activity levels by which the ordinary activity levels

* See Sraffa § 70.

e have to be augmented:

$$e' + q' = (s^1 + \delta_1 e_1)(B-A)^{-1} = e' + \delta_1 e_1 (B-A)^{-1}$$

(δ_1 should be chosen sufficiently small for $e + q \geq 0$). As stated in the sub-systems approach, we may interpret $q'l$ as the additional labour which is consumed in the production of s_1 . We find:

$$q'l = \delta_1 e_1 (B-A)^{-1} l = \delta_1 e_1 u = \delta_1 u < 0,$$

that is to say, we find that the increase in the production of good 1 does not require an increase in the consumption of productive labour, but that, on the contrary, it seems to have set labour free.

Such a state of affairs looks very paradoxical at first sight. We shall see, however, that not only does it occur frequently, but even that it is typical for a certain type of inefficient machinery (see §§ 15, 16).

Suffice it to say at present that negative prices at $r = 0$ are perfectly compatible with positive prices for some $r > 0$ which conceal the inefficiency implied in negative labour values: an economy working at $r = 0$ would not indulge in producing good one in ever increasing quantities simply for the reason that this could be done without engaging more labour, but it would switch to new techniques, which allow positive $p(0)$. (In the case of fixed capital the good in question may well be produced in even greater quantities after the switch, but as a waste product, not a commodity; thus outside the system).

5. Distribution and the Standard Commodity

Assuming the subsistence wage of the workers is not already implicit in the input matrix A, we introduce it as follows: if labour is unskilled and interchangeable between industries, it is consistent to regard the labour force as homogenous in its consumption pattern. Assume the whole wage is for subsistence. If the wage rate is uniform, the workers in every industry will buy a fraction of the total wage $d = (d_1, \dots, d_n)$ proportional to their number or (what amounts to the same thing) proportional to the labour time necessary in that industry so that the matrix

$$D = l d' = \begin{bmatrix} l_1 d_1, \dots, l_1 d_n \\ l_2 d_1, \dots, l_2 d_n \end{bmatrix}$$

represents the real wage consumed by the workers in every industry.

It depends on the existence of a uniform wage rate but not of a uniform rate of profit.

$\ell_w = D u$, $u = (\mathbb{R} - A)^{-1} l$, is then the labour time the workers spend in each industry on earning their living. From

$$\ell_w = (l d') u = l (d' u) = \alpha l$$

$$\alpha = d' u \leq s' u = e^t (\mathbb{B} - A) (\mathbb{B} - A)^{-1} l = 1$$

it follows that the time they have to work for themselves is proportional to the time they spend working for the capitalists - that is to say, Marx's "rate of surplus value" $\frac{1-\alpha}{\alpha}$ is uniform irrespective of the rates of profit. (See Morishima, Okishio / 7, 10 / etc.)

The rate of surplus value is a concrete measure for income distribution, but Mr. Sraffa chooses another one, partly because the wage, as soon as it exceeds the subsistence level, cannot be determined before prices are known, partly because he considers the rate of profit as the independent variable, and regards it as previous to the distribution of physical income. We refer to his book for the detailed discussion of his standard of value (the "standard commodity") and list only briefly its main features.

To make an easy start, we postulate the polynomial

$$\det(\mathbb{R} - (1+r)A) = 0$$

has one and only one simple root $R > 0$ in $\{r \mid r \geq 0\}$. The real "eigen-vector" q

$$(1+r) q' A = q' B$$

associated with it, is normalized by

$$q' l = 1^*$$

q are the "standard multipliers", $t^i = q^i (\mathbb{B} - A)$ the standard

* Remember $e^t l = 1$. We shall tacitly assume from now on (in line with the conclusions of §3 and 7) that $q' l \neq 0$.

commodity*. We use it to define standard prices

$$P_t = \frac{(1-(1+r)A)^{-1}l}{t^1(B-A)Al}, t^1 P_t = 1.$$

The relation

$$\boxed{A = \frac{r}{R} + w_t}$$

which follows from

$$1 = q^1(B-A)P_t(r) = q^1[rA P_t(r) + w_t l]$$

$$= \frac{r}{R} q^1(B-A)P_t(r) + w_t$$

has become very famous. Because

$$\frac{r}{R} = \frac{-q^1 A p}{q^1(B-A)p},$$

it shows how the shares of profits and wages (wages are equal to the wage rate for $e^1 l = 1$) vary linearly in an economy in standard proportions, that is, in an economy for which $q = e$ (e is always assumed to denote the actual proportions).

If $e \neq q$, $1 = \frac{r}{R} + w_t$ still holds as a relation between the rate of profit r and the wage rate;

$$w_t = A - \frac{r}{R} = \frac{1}{t^1 \hat{p}(r)}.$$

The real wage may consist of any basket of goods with $w_t = \bar{a}^1 p_t$, $\bar{a}^1 \leq e^1(B-A)$. If the real wage is a fraction $\lambda(r)$ of a given basket \bar{a} in some range of r and if $\hat{p}(r)$ (prices in terms of the wage rate) rise monotonically with r, $\lambda(r)$ will fall, whatever \bar{a} .

If some $\hat{p}_i(r)$ do not rise with r, the value of the wage $w_t = \lambda(r) \bar{a}^1 p_t(r)$ will fall with r, but not necessarily $\lambda(r)$. (See §1.)

The standard prices

$$P_t(r) = (1 - \frac{r}{R})(B - (1+r)A)^{-1}l = (1 - \frac{r}{R})\hat{p}(r)$$

* $t^1 \geq 0$, if $q^1 \geq 0$; for $t^1 = q^1(B-A) = R q^1 A$.

tend for $r \rightarrow 0$ to labour values

$$p_t(0) = (R - A)^{-1} \ell = u^*.$$

They make the value of the national product at $r = 0$ equal to one (equal to the value of the standard commodity):

$$e^t(R-A)p_t(0) = e^t(R-A)u = e^t\ell = 1$$

and possess, unlike $\hat{p}(r)$, a limit for $r = R$; for the limit

$$\pi = \lim_{r \rightarrow R} p_t(r) = \lim_{r \rightarrow R} \left[\frac{(R-r)(B-(1+r)A)\ell}{R \det(B-(1+r)A)} \right]$$

exists, because R is a simple root of $\det(B - (1+r)A) = 0$, and it is clear that $\pi = p_t(R)$, with $p_t(R)$ defined by

$$(1+R)A p_t(R) = B p_t(R), \quad t^1 p_t(R) = 1.$$

The real wage a ($a^t p_t(r) = w_t$) is zero at $r = R$, whatever $a \geq 0$, if $p_t(R) > 0$.

6. Expansions of the Price Vector

We continue to assume $\det(B - (1+r)A) = 0$ has a simple positive root. The smallest is denoted by R . Moreover we assume (A, B) is basic.

In single product systems the standard prices could be reduced to series of "dated quantities of labour":

$$p_t(r) = \left(1 - \frac{r}{R}\right) (I - (1+r)A)^{-1} \ell \quad (6.1)$$

$$= \left(1 - \frac{r}{R}\right) (\ell + (1+r)A\ell + (1+r)^2 A^2 \ell + \dots).$$

In § 4 the $A^k \ell$ were recognized as dated labour terms.

$p_t(r)$ can be constant, therefore equal to values, if and only if

$$(1+r)A(B-A)^{-1}\ell + \left(1 - \frac{r}{R}\right)\ell = B(B-A)^{-1}\ell;$$

therefore if and only if

$$RA(B-A)^{-1}\ell = \ell.$$

The expansion ($\det B \neq 0$!)

$$\begin{aligned} p_r(r) &= \left(1 - \frac{r}{R}\right) (B - (1+r)A)^{-1} l \\ &= \left(1 - \frac{r}{R}\right) (I - (1+r)B^{-1}A)^{-1} B^{-1} l \\ &= \left(1 - \frac{r}{R}\right) (I + (1+r)B^{-1}A + (1+r)^2(B^{-1}A)^2 + \dots) B^{-1} l \end{aligned} \quad (6.2)$$

(where it is admissible) represents a reduction to dated quantities of labour, for joint production systems. We shall also use

$$\begin{aligned} p(r) &= (B - (1+r)A)^{-1} l \\ &= (I - r(B-A)^{-1}A)^{-1} (B-A)^{-1} l \\ &= (I + r(B-A)^{-1}A + r^2((B-A)^{-1}A)^2 + \dots) u. \end{aligned} \quad (6.3)$$

Theorem 6.1

The series (6.2), [(6.3)], converges for $0 \leq r < R$, if $B^{-1}A \geq 0$ [$(B-A)^{-1}A \geq 0$]. If $B^{-1}A \not\geq 0$ [$(B-A)^{-1}A \not\geq 0$], it converges, if at all, in no larger neighbourhood of zero than $0 \leq r < R$.

Proof: We have to see whether a positive limit to r exists so that the absolute values of all the complex eigenvalues of $(1+r)C = (1+r)B^{-1}A$ are smaller than one. If a positive limit exists, (6.2) will be valid within that range.

Replace the elements of $C = B^{-1}A$ by their absolute values and denote the ensuing semipositive matrix by $|C|$. $|C|$ is indecomposable (Theorem 2.1) and has a maximal positive eigenvalue λ with which $q > 0$, $q^t |C| = \lambda q^t$ is associated. Let μ, x be any (complex) eigenvalue and eigen-vector of $(1+r)C$:

$$(1+r)Cx = \mu x.$$

$|x|$ is the vector of absolute values of the components of x . From

$$|\mu x| = |\mu| |x| = (1+r) |C x| \leq (1+r) |C| |x|$$

$$|\mu| q^t |x| \leq (1+r) q^t |C| |x| = (1+r) \lambda q^t |x|$$

and $q > 0, x \geq 0$, it follows

$$|\mu| \leq (1+r)\lambda.$$

Thus, $|\mu| < 1$ if $(1+r)\lambda < 1$. Distinguish:

- a) $B^{-1}A \geq 0$. Then $B^{-1}A = \begin{vmatrix} B^{-1}A \end{vmatrix}$, $q^T B^{-1}A = \lambda q^T$ and $B^{-1}A p = \lambda p$ with $p > 0$. From Lemma, §2, we have $\lambda < 1$.

According to the Frobenius theorem, λ is the greatest real root of $B^{-1}A$. R was defined as the smallest real root of $\det(B - (1+r)A) = 0$.

Thus $\lambda = \frac{1}{1+R}$ which proves the first assertion.

- b) $B^{-1}A \not\geq 0$. The series converges for $0 \leq r \leq \frac{1}{\lambda} - 1$, if $\lambda < 1$. This may be the case, but since $\frac{1}{1+R}$ is one of the eigenvalues of $B^{-1}A$, we have from $|\mu| \leq (1+r)\lambda$ for $r = 0$ $\frac{1}{1+R} \leq \lambda$.

The proof for (6.3) is analogous.

q.e.d.

III Maximum Rate of Profit and Standard Commodity in General Joint Production Systems.

7. Peculiarities of Joint Production Systems

The questions of the existence and uniqueness of the standard commodity are mathematically and in economic interpretation intricately linked to the notion of basics. Firstly, because the characteristic polynomial of a non-basic system falls apart

$$\begin{aligned} \det(B - (1+r)A) &= \det M \det(B - (1+r)A) = \det(MB - (1+r)MA) \\ &= \det(B_1^2 - HB_2^2 - (1+r)(A_1^2 - HA_2^2)) \det(B_2^2 - (1+r)A_2^2). \end{aligned}$$

Roots of the second part of this equation (roots of $\det(B_2^2 - (1+r)A_2^2) = 0$) yield eigenvectors totally irrelevant from the economic point of view, if the interpretation of non basics given in §2 is accepted. This is obvious, if a root R of $\det(B_2^2 - (1+r)A_2^2) = 0$ is greater than the maximum rate of profit R in the basic system (which we here suppose to exist): prices cannot rise beyond R , because $\hat{p}_1(R)$ diverges. In the opposite case (see Sraffa, Appendix B) all one can say is that a uniform rate of profit for all processes A_2, B_2, \dots is impossible

at or around $r = \tilde{R}$; for

$$(\bar{B}_2^2 - (1+r) \bar{A}_2^2) \hat{p}_2 = \lambda_2 - (\bar{B}_2^1 - (1+r) \bar{A}_2^1) \hat{p}_1$$

is not solvable for \hat{p}_2 at $r = \tilde{R}$. Thus, if r is to remain uniform, some non-basic processes have to be replaced.

We now turn to roots of $\det(\bar{B}_1^1 - (1+r) \bar{A}_1^1) = 0$. If there are several positive roots, we define the smallest of them to be the relevant maximum rate of profit for the intuitive reasons given in Sraffa, § 64. The question is whether it exists and whether the standard commodity associated with it is unique.

In order to solve the problem of uniqueness we have again to recall our discussion of the basic system. Although we argued there that multiple eigenvalues tend to be exceptional, we propose here the following solution for the sake of completeness:

If the smallest positive root of $\det(\bar{B}_1^1 - (1+r) \bar{A}_1^1) = 0$ is semisimple*, only one eigenvector and therefore a uniquely defined standard commodity exist. If the root is not semisimple, it follows from theorem 3.2 in § 3 that some $\bar{q}, \bar{q} \neq 0$ with $\bar{q} \hat{p}(r) = 0$ exist. Suppose $\bar{q} = (\bar{q}_1, \dots, \bar{q}_n), \bar{q}_i \neq 0$. The column vectors of \bar{A}_1^1 are denoted by \bar{a}^j : $\bar{A}_1^1 = [\bar{a}^1, \dots, \bar{a}^n]$, $\bar{a}^j = (a_{1j}^j, \dots, a_{nj}^j)^T$, $j = 1 \dots n$ (there are m non-basics). Define

$$Y = \frac{1}{\bar{q}_i} \begin{bmatrix} a_{1i}^1 q_1, \dots, a_{ni}^1 q_n \\ a_{1i}^2 q_1, \dots, a_{ni}^2 q_n \end{bmatrix}$$

Since $Y \hat{p}(r) = 0$, we find, that the system

$$(\bar{A}_1^1 - Y, \bar{B}_1^1, \bar{\lambda}_1)$$

has the same prices as $(\bar{A}_1^1, \bar{B}_1^1, \bar{\lambda}_1)$ for all rates of profit, but it differs from our original system in that $(\bar{A}_1^1 - Y, \bar{B}_1^1, \bar{\lambda}_1)$ is non-basic because the i -th column is zero. This non-basic will either have to be eliminated and the procedure if necessary repeated until the disturbing multiple roots have disappeared. Or else, if this extension of the concept of a basic system seems to lead too far away from economic intuition, this sudden and unexpected

* R with $\det(B - (1+R)A) = 0$ is called semisimple if $\text{rk}(B - (1+R)A) = n-1$ (see § 3).

appearance of a non-basic will induce one to rule out a priori all systems with non semi-simple roots of $\det(B - (1+r)A) = 0$.

We conclude at any rate that we either have a maximum rate of profit with which a unique standard commodity is associated or no maximum rate of profit and no standard commodity at all. Roots with several eigenvectors or ambiguities about which of several roots should be chosen, may be ruled out.

Our sole concern in §§ 8 and 9 in the first part of this paper will be to find sufficient conditions for the existence of the maximum rate of profit in the basic system having excluded all sorts of pseudo maximum rates of profit. Such conditions by no means spring directly from the concept of a basic system as is the case with single product systems. Nor is there a comprehensive and straightforward mathematical theory.

We have to specify joint production systems by economic criteria, if we want to get anywhere. The specifications are ~~restrictive~~ in relation to our discussion of the basic system. E.g. we assume throughout that the matrices of the basic system are semi-positive.

Before we start, we prove a trivial, but nevertheless very important theorem. We show that no economic system are prices positive and finite for all rates of profit. If $\hat{p}(r) > 0$ at $r = r_0$ and the rate of profit rises, there must come a point $r = r_A$ at which either prices in terms of the wage rate become infinite (the wage falls to zero, $r_A = R = \text{maximum rate of profit}$). Or some prices turn negative. Systems without a maximum rate of profit are not absurd because in them, too, there is a limit to the rise in the rate of profit and the viability of the system, in the shape of negative prices. Systems with a maximum rate of profit (and standard commodity) are on the other hand not immune from negative prices.

Theorem 7.1

Let (A, B, ℓ) be a basic system, $A \geq 0$, $B \geq 0$. Either a maximum rate of profit and a standard commodity exist and/or prices in any standard do not remain positive for all positive r .

Proof: Suppose $\hat{p}(r) = (B - (1+r)A)^{-1} \ell \geq 0$, $\det(B - (1+r)A) \neq 0$

for $r \geq r_0 \geq 0$. Since $e^r A > 0$ (according to the Lemma in § 2) and since $e^r B = e^r$ (the usual normalization) and since $e^r (B-A) \geq 0$, we can write

$$e^r A = \gamma e^r + f^r, \quad f^r \geq 0, \quad 0 < \gamma < 1,$$

and get for all $r \geq r_0 \geq 0$

$$\begin{aligned} B\hat{p}(r) - (1+r)A\hat{p}(r) &= \ell \\ e^r B\hat{p}(r) - (1+r)e^r A\hat{p}(r) &= \\ &= e^r \hat{p}(r) - (1+r)\gamma e^r \hat{p}(r) - (1+r)f^r \hat{p}(r) \\ &= e^r \hat{p}(r)(1 - (1+r)\gamma) - (1+r)f^r \hat{p}(r) = e^r \ell = 1. \end{aligned}$$

Since $e^r \hat{p}(r)$ and $f^r \hat{p}(r)$ remain positive for all r , the left hand side of the last equation must have turned negative for $r > \frac{1}{\gamma} - 1$ which is a contradiction. Either or both of the assumptions must therefore be false. The extension to other price standards is immediate.

q.e.d.

In particular, we conclude: if prices $\hat{p}(r)$ (prices in terms of the wage-rate) do not fall as r rises from some r_0 with $\hat{p}(r_0) > 0$, they must diverge to infinity at some finite rate of profit which is the maximum rate of profit.

8. All-productive and Related Systems

We transform (A, B, ℓ) as in § 6:

$$\hat{p}(r) = (B - (1+r)A)^{-1}\ell = (I - r(B-A)^{-1}A)^{-1}(B-A)^{-1}\ell.$$

If (A, B) is all productive ($(B-A)^{-1} \geq 0$), it has all the essential properties of a single product system:

If (A, B) is basic, $(B-A)^{-1}A$ is indecomposable. There is a simple root $R > 0$ of $\det(I - r(B-A)^{-1}A) = 0$ which is also a simple root of $\det(B - (1+r)A) = 0$ so that the eigenvector associated with it is positive:

$$(1+r)A\hat{p} = B\hat{p}, \quad p > 0,$$

because $(B-A)^{-1}A$ is indecomposable and

$$p = R(B-A)^{-1}A\hat{p}.$$

The standard multipliers

$$(A + R)qA = qB$$

and the standard commodity $t' = q'(B - A) = Rq'A$
are at least semipositive.

From the expansion of the pricevector in § it is seen that all-productive systems have positive prices for $0 \leq r < R$. They are also alone in being immune against the inefficiency mentioned in § which is connected with negative labour values.* Unfortunately, all-productive systems are as such quite untypical for joint production: in a two sector model where two goods are produced by two processes it is necessary and sufficient for all-productivity that each process produces more of one commodity than it uses and uses more of the other than it produces. More convincing examples exist for $n > 2$, but one does not get very far.

It is however, possible to find some meaningful generalizations from all-productive systems:

1. $(B - (1+\bar{r})A)^{-1} \geq 0$ for some $\bar{r} \geq 0$. $(B - (1+\bar{r})A)^{-1} \geq 0$ for some $\bar{r} \geq 0$ is a condition considerably weaker than $(B-A)^{-1} \geq 0$. Economically, it means that no price $\hat{p}_j(r)$ in terms of the wage rate falls as a result of any partial increase of the wage $w \rightarrow w + \delta w_j$ or of working hours $\ell_j \rightarrow \ell_j + \delta \ell_j$ in any one industry j (see example in § 11). The system $((1+\bar{r})A, B, \ell)$ fulfills the same conditions as an all-productive system. In particular, maximum rate of profit and standard commodity exist and prices $p_j(r)$ are positive for $\bar{r} \leq r \leq R$.

2. $A(B-A)^{-1} \geq 0$. If $A(B-A)^{-1}$ is positive and indecomposable, a maximum rate of profit and positive standard multipliers exist, for

* Positive $u = \hat{p}(0) = p_t(0) = (B-A)^{-1}\ell$ does not guarantee positive prices, but for every A, B with $R(A-B)^{-1}\ell = p$ for some $p > 0$ the irregular system $(A, B, \ell = p)$ has positive prices (constant organic composition of capital) for $0 \leq r \leq R$.

$$q' = R q' A (B-A)^{-1},$$

$$(A+R) q' A = q' B.$$

Because if $A(B-A)^{-1} = Q \geq 0,$

$$A = Q(B-A), Q \geq 0,$$

$$a_i = q_i(B-A), q_i \geq 0,$$

$$Q = [q_1, \dots, q_n]',$$

This condition means that all goods can be produced independently in the proportions in which they are required as inputs for production, if constant returns to scale prevail. Again, this condition is weaker than $(B-A)^{-1} \geq 0$, for it does not imply that all positive net outputs are producible at positive activity-levels; it implies only that all convex combinations of the vectors a_1, \dots, a_n are producible.

If $q_i = a_i (B-A)^{-1} \geq 0, i=1, \dots, n$, the production of the commodities a_i required as inputs in process i for the production of output b_i can be increased everything else remaining equal. Thus, although a system with $(B-A)^{-1} \geq 0$ is not so flexible as to be able to produce a net output of commodities in all conceivable combinations, it is at least capable of expanding and contracting all lines of production independently of each other.*

9. The Standard Commodity and the Possibility of Balanced Growth

Consider A, B as iron Neumann matrices (assuming that constant returns to scale prevail) and assume that A includes the subsistence wage of the workers so that we may conceive of an actual rise in the rate of profit up to what corresponds to the maximum rate of reproduction in a state of balanced growth.

The maximum rate of balanced growth α cannot be pushed up indefinitely; there is an upper limit to the α for which

$$\alpha q' A \geq q' B, q \geq 0,$$

*Such a condition could play a part in a dynamical system which does not pre-suppose a uniform rate of profit, but shows how it comes about. One of the mechanisms involved in the process of equalisation is the free flow of circulating capital.

the condition of expanded reproduction. Von Neumann and his followers have shown how a semi-positive pricevector can be associated with the maximum rate of growth

$$\alpha q^T A \geq q^T B, \quad q \geq 0,$$

so that

$$\alpha A p \leq B p, \quad p \geq 0,$$

and so that

$$\alpha q^T A p = q^T B p$$

i.e. so that overproduced goods fetch zero prices and unprofitable activities are at a standstill. (Profits are here at most "normal" profits, the rate of profit is the maximum rate of profit and equals the rate of balanced growth).

These α , q , and p must exist for our system A , B ($A \geq 0$, $B \geq 0$, $a_i \neq 0$, $b_i^T \neq 0$, $i = 1, \dots, n$)*, since all the assumptions conventionally used to prove von Neumann's theorem are fulfilled.

I) If α is smaller or equal to one, A , B is inefficient because it is incapable of expanded reproduction in a state of balanced growth.

II) If $\alpha > 1$ and the strict inequality holds in either $\alpha q^T A > q^T B$ and/or $\alpha A p \leq B p$ in some component, q and p cannot both be positive for $\alpha q^T A p = q^T B p$. Thus, some activities must be unprofitable at that rate of profit and/or there are over-produced goods.^{**}

* These assumptions were stated in §1. They are sufficient for von Neumann according to Kemeny Morgenstern - Thompson / 4 /. For the reasons given in §2 we suppose A , B are basic.

** A striking example of case II is given in [17] in the chapter on Fixed Capital where the standard commodity t exists for some $r = R$, but with $p_t(R) \neq 0$. The maximum rate of growth cannot be equal to $1+R$. It turns out that α can be increased beyond $1+R$ by abandoning processes using old equipment so that the old machines appear as overproduced goods and processes using them as uneconomic.

If both these inefficiencies (I and II) are ruled out, we have

$$\alpha q' A = q' B, \quad \alpha = 1 + R, \quad R > 0, \quad (1+R) A p = B p. \quad q$$

may be interpreted as standard multipliers. Thus, $t' = q' (\mathbb{I} - A)$ the standard commodity, exists and is, together with $p_t(R)$
 $(1+R) A p_t = B p_t$, semi positive.

It is possible to give a more specific sufficient condition for this:

Gale* has shown that the maximum rate of expansion can be defined as

$$\alpha = \max_{q \geq 0} \min_j \frac{q' b_j}{q' a_j} \quad (A = [a_1^t, \dots, a_n^t],$$

$$B = [b_1^t, \dots, b_n^t]).$$

$\alpha(q) = \min_j \frac{q' b_j}{q' a_j}$ is a continuous function on the compact set $\{E \mid q \geq 0, q^t e = 1\}$. Balanced growth occurs at the point \bar{q} where $\alpha(\bar{q})$ attains its maximum on E. If there are uneconomic activities, the maximum $\alpha(\bar{q})$ is attained on the boundary of E. If there is overproduction, $\min_j \frac{\bar{q}' b_j}{\bar{q}' a_j}$ will be smaller than some $\frac{\bar{q}' b_j}{\bar{q}' a_j}$ (the overproduced goods).

The latter is not the case, the highest rate of reproduction for each good is found in a different process so that there is a one-to-one correspondence between the n processes and the n goods in terms of the highest rate of reproduction. Each process will then contribute for some good to the attaining of the maximum rate of growth.

The conditions necessary for balanced growth without over-production are slightly more stringent:

Theorem 9.1

If permutations of goods and processes and a constant α can be found so that for all $a_i \neq 0$

$$0 \leq \frac{b_i^j}{a_i^j} < \alpha < \frac{b_j^i}{a_j^i} \leq \infty$$

and so that $b_i^j = 0$ for $a_i^j = 0$, $j \neq i$, it follows that a maximum rate of profit R and a semipositive standard commodity t exist

Note:

The assumption

$$0 < \frac{b_{ij}}{a_{ij}} < \alpha < \frac{b_{ij}}{a_{ij}} \quad (i \neq j)$$

is necessary if only

$$\frac{b_{ij}}{a_{ij}} < \frac{b_{ij}}{a_{ij}} \quad \text{is}$$

assumed, overproduction is not ruled out, as examples with $n \geq 3$ show. For $n = 2$ however, the assumption is sufficient (see also [1]).

The proof of Theorem 9.1 may either be based on von Neumann's theorem, using Gale's method* or we may proceed as follows:

$B - \alpha A$ has positive elements in the diagonal and negative elements outside. After premultiplying A , B from the right with the diagonal matrix

$$D = (d_{ij}) = \left(\begin{array}{ll} 1 & d_{ij} \\ b_{ij} - \alpha a_{ij} & \end{array} \right), \quad d_{ii} = 1, \quad d_{ij} = 0 \quad (i \neq j),$$

we can decompose $BD - \alpha AD$ into

$$BD - \alpha AD = I - C$$

with $C \geq 0$. Thus, there is $\bar{\alpha} > 0$, $p \geq 0$, so that $\bar{B}Dp = \bar{\alpha}ADp$.

From the Lemma in [2] $\bar{\alpha} > 1$. The rest is obvious,

q.e.d.

Theorem 9.1 could be generalized by using more sophisticated assumptions, but one does not seem to gain anything significant: the main insight gained from this paragraph is the one derived from the von Neumann model: systems without a semi-positive standard commodity and semipositive $p_t(R)$ suffer from specific inefficiencies revealed at $r = R$, much in the same way as systems which are not all-productive are susceptible to suffer from inefficiencies revealed at $r = 0$, but covered at $r = \infty > 0$. Both these inefficiencies occur only in systems where the possibility of substitution is inherent: they are not all-engaging and yet basic.

Although such inefficiencies may not be characteristic for general joint production systems and on this assumption the existence of a maximum rate of profit and a standard commodity is assured for general joint production systems (compare the coal-coke-gas-example in [2]), these inefficiencies turn out to be typical for fixed capital which

* see Gale / 2 /.

we shall therefore have to approach by mathematically different methods.

IV Graphic Techniques

10. Method

The patient reader will probably be longing for concrete examples. In order to avoid lengthy calculations they ^{will} be given graphically.

The system $P_t = w(B - (1+r)A)^{-1}l$ has 21 fixed and 5 variable coefficients for $n = 3$. A proper graphic representation is therefore not trivial.

The common procedure is to take a_i , b_i as vectors and to interpret $a_i p$ etc. as scalar products. While this approach is more satisfactory from the point of view of linear and multilinear algebra, it proves less cumbersome to draw the column vectors a^i, b^i of the goods and of labour on axes which represent the processes.

The measure of goods is given by the familiar normalization

$$e^i B = e^i (b^1, \dots, b^n) = e^i (e = (1, \dots, 1)).$$

Instead of standard prices P_t or prices in terms of the wage-rate \hat{p} we use unit sum-prices P_e :

$$P_e = \frac{(B - (1+r)A)^{-1}l}{e^i (B - (1+r)A)^{-1}l} = w_e \hat{p}(r)$$

$$w_e = \frac{1}{e^i (B - (1+r)A)^{-1}l} = \frac{1}{e^i \hat{p}(r)}, \quad e^i P_e = 1.$$

We interpret $B P_e$, $A P_e$, $(B - (1+r)A) P_e$ as the weighted sum of the vectors b^1, \dots, b^n , where $[b^1, \dots, b^n] = B$, a^1, \dots, a^n where $[a^1, \dots, a^n] = A$ and $b^1 - (1+r)a^1, \dots, b^n - (1+r)a^n$ where $[b^1 - (1+r)a^1, \dots, b^n - (1+r)a^n] = B - (1+r)A$. The set

$\{x \mid x = A P_e, e^i P_e = 1\}$
is the simplex spanned by $[a^1, \dots, a^n]$.

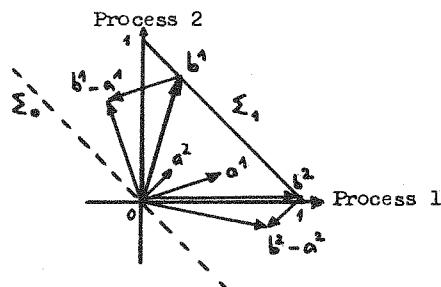
The conditions $e^i B = e^i, e^i l = 1, B \geq 0, l \geq 0$ mean the b^1, \dots, b^n, l are points on the simplex

$$\Sigma_A = \{x \mid e^i x = 1, x \geq 0\},$$

$e^i(B-A) > 0$ means the $b^i - a^i$ are
"to the right of and above"

$$\Sigma_0 = \{x \mid e^i x = 0\}.$$

For a two commodities/two processes system the diagram looks as follows:



The set

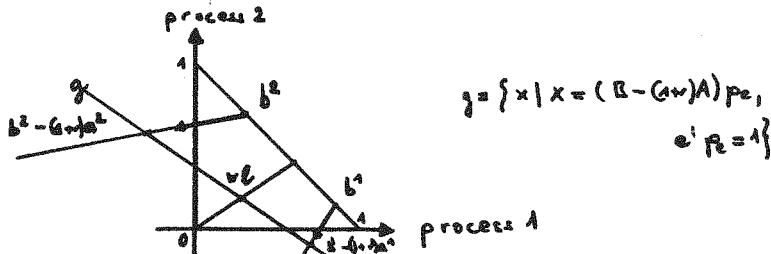
$$\{x \mid x = (b^1 - (1+r)a^1)p_1 + (b^2 - (1+r)a^2)p_2, p_1 + p_2 = 1\}$$

$$= \{x \mid x = (B - (1+r)A)p_e, e^i p_e = 1\}$$

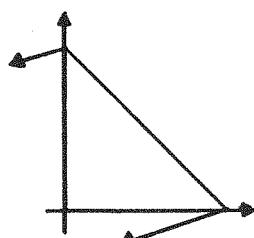
is represented by the straight line through $(b^1 - (1+r)a^1), (b^2 - (1+r)a^2)$; the solution of the equation

$$(B - (1+r)A)p_e = w_e l, \quad e^i p_e = 1,$$

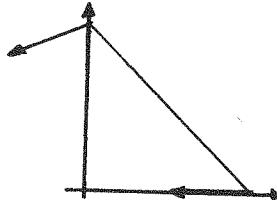
is represented by the intersection of that line with l in the point $w_e l$ (no lines for b^1 nor a^1 are drawn, only $b^1 - a^1$ as the vector pointing from b^1 to $b^1 - a^1$):



The following three examples are:

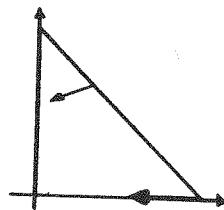


a basic single product system,



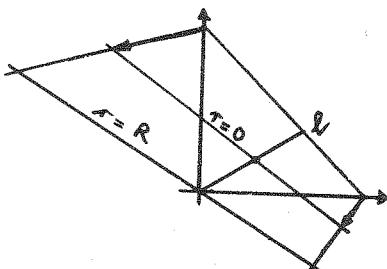
a nonbasic
single product
system,

and

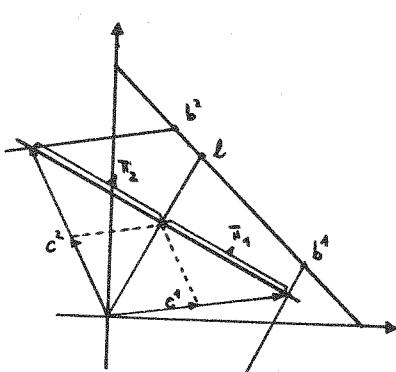


a nonbasic
joint production
system.

Normally, as r increases, w diminishes until $r = R$ and $w_e = 0$ are reached and $b^1 - (1+r)a^1$, $b^2 - (1+r)a^2$ are linearly dependent:



Price ratios are, of course, represented in the diagram. How is shown in the following two diagrams:



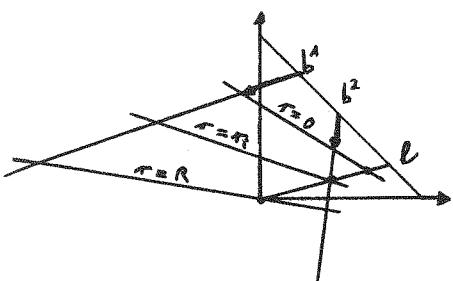
$$c^1 = (b^1 - (1+r)a^1)p_1$$

$$c^2 = (b^2 - (1+r)a^2)p_2$$

One reads from the diagram:

$$\frac{r_1}{r_2} = \frac{p_2}{p_1}$$

(Note the inverse relationship).

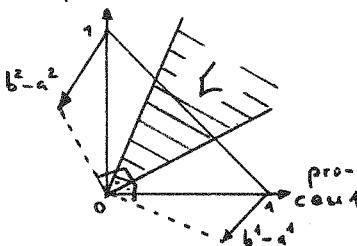


In this example, one price is negative for $r = 0$ and small r , then both prices are positive (beyond $r = r_i$). (It is geometrically evident this cannot occur in single product systems.)

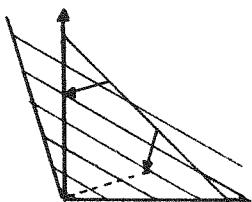
We now turn to activity levels. We represent them as vectors.

$q^i b^i$ (q activity level, b^i vector of outputs in commodity i) is interpreted as a scalar product.

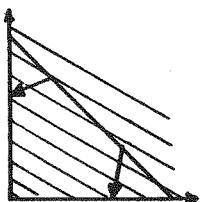
process 2



The shaded cone L is the set of all activity levels for which $c^i = q^i(B - A) = (q^i(b^i - a^i), q^i(b^2 - a^2)) \geq 0$. It contains $e = (1, 1)^T$

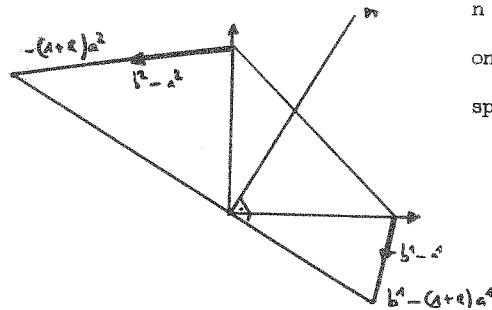


$(B - A)^T > 0$ means that L lies in the (strictly) positive orthant. Here, this is not the case.



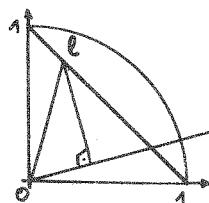
Here, A, B is basic, $(B - A)^{-1} \geq 0$, but not $(B - A)^{-1} > 0$. (Evidently, this cannot happen with single product systems).

The standard multipliers form a vector q orthogonal on each of the vectors $b^i - (1 + R)a^i$, because $q^i((B - (1 + R)A)a^i) = 0$. Thus, q is orthogonal on the hyperplane, containing the origin and spanned by those vectors $b^i - (1 + R)a^i$, $i = 1, 2$.



n is the normal
on the hyperplane
spanned by

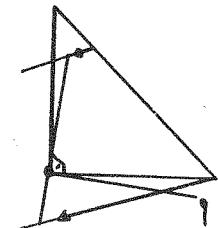
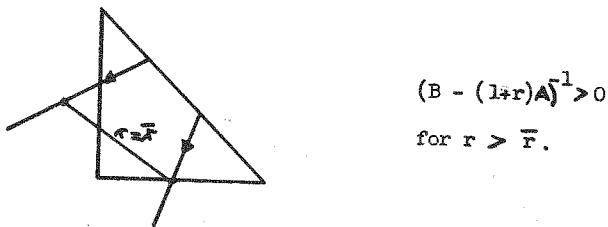
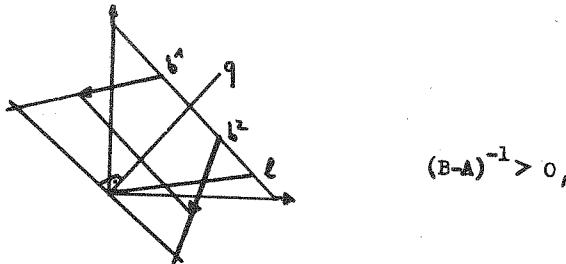
$$b^1 - (a+r)a^j, j=1,2.$$



The length of q is determined
by $q^1 l = 1$. q can be obtained
geometrically by mirroring the
projection of l on n at the unit
circle.

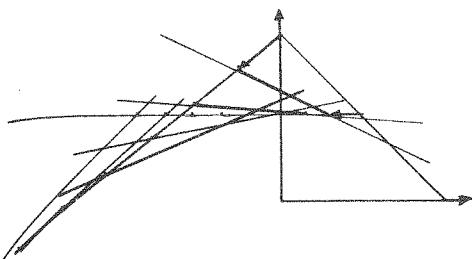
11. Applications

It is now evident from the geometry that $p_e(r) > 0$, $p_e(R) > 0$, $q > 0$
for all single product systems and $q > 0$, $p_e(R) > 0$ and $p_e(r) > 0$
for $r > \bar{r}$ for systems with $(B - (1+r)A)^{-1} > 0$:



The standard multipliers may
not be positive

and may not exist at all:



This corresponds basically to Manara's example, greatly exaggerated.

To "show" the standard multipliers which do not exist, one draws the straight lines

$$\{x \mid x = (B - (1+r)A)p, e^T p = 1\}$$

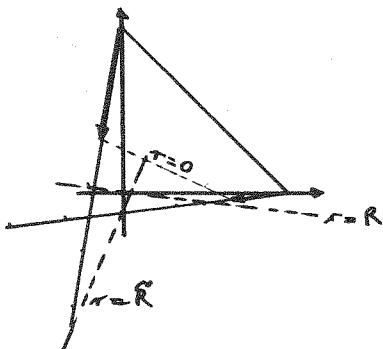
for "many" rates of profit. The straight lines cover the plane up to an area bounded by a curve of second degree. If one of these lines hits the origin for $r = \bar{R}$, it follows that $b^1 - (1+\bar{r})a^1$ and $b^2 - (1+\bar{r})a^2$

are linearly dependent for $r = \bar{R}$.

If the origin happens to lie in the area which is left free by the lines

$$\{x \mid x = (B - (1+r)A)p, e^T p = 1\},$$

no standard multipliers exist, nor does a standard commodity.



In general, there are two eigenvectors in a two sector model, even in single product systems. But only one of them has positive components: the one corresponding to the lower rate of profit $r = R$. The eigenvalue $r = \tilde{R}$ has no economic meaning.

Many other special cases could be considered. In particular, the irregular systems can be represented, but we leave it to the reader.

Only one inefficiency is worth dealing with in particular: the one which corresponds to theorem 9.1 in [9].

Consider (it makes sense for $n = 2$) systems with $A > 0$.

$$g_{ij} = \frac{b_j^1}{a_i^1}$$

is the rate of reproduction of good j in process i
($i, j = 1, 2$).

We define: a system involving two goods and two processes is inefficient, if one process is better than the other for both

goods, i.e. if

$$(g_i^1, g_i^2) > (g_j^1, g_j^2) \text{ for } (i,j) = (1,2) \text{ or } (2,1).$$

The example proposed by Mr. Manara is inefficient in this sense.

Theorem

If the inefficiency above is ruled out, a positive standard ratio R and a positive q and p (R) exist.

Proof: If the system is efficient, we have, without loss of generality,

$$g_1^1 > g_2^1 \text{ and } g_1^2 < g_2^2$$

or

$$\frac{b_1^1}{a_1^1} > \frac{b_2^1}{a_2^1} \text{ and } \frac{b_1^2}{a_1^2} < \frac{b_2^2}{a_2^2}$$

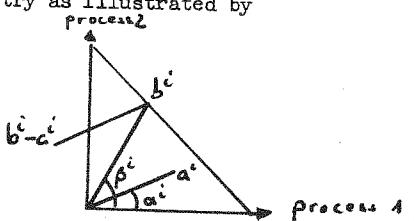
or

$$\frac{b_1^1}{b_2^1} > \frac{a_1^1}{a_2^1} \text{ and } \frac{b_1^2}{b_2^2} < \frac{a_1^2}{a_2^2}.$$

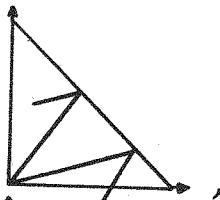
Now

$$\frac{b_1^i}{b_2^i} = \operatorname{ctg} \beta^i, \quad \frac{a_1^i}{a_2^i} = \operatorname{ctg} \alpha^i$$

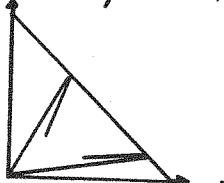
where β^i, α^i are the angles between the axes representing the first process and vector b^i, a^i . The rest is evident from the geometry as illustrated by



drawn from

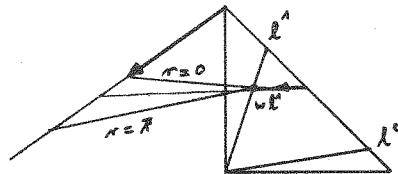


and from



q.e.d.

Such inefficient systems may nevertheless possess positive prices for low r , if the labourvector is appropriate^{*}.



If $\lambda = \lambda^1$, prices are positive for $0 \leq r < \bar{r}$. If $\lambda = \lambda^2$, one price is negative for all r .

Three-dimensional examples will be considered later.

* That is to say, if the less efficient technique employs less labour.

PART TWO: SPECIFIC JOINT PRODUCTION SYSTEMS

I. Fixed Capital Systems

12. Definition of a Pure Fixed Capital System

As the reader will remember, Mr. Sruffa concentrates in his Chapter X, on "Fixed Capital", on the changes in value of single machines of different vintages. Taking that analysis for granted we inquire into the interaction between fixed capital and the economy as a whole, confining ourselves to "pure fixed capital systems."

In order to facilitate the discussion, we introduce the following terminology: in any fixed capital system it shall always be possible to distinguish between finished goods or products (circulating capital and new machines) and intermediate goods or products (old machines of various vintages). In the normal case, corresponding to § 76 - 78 in "Production of Commodities by Means of Commodities", each process produces one and only one finished good (no superimposed joint production) together with one, several or possibly no intermediate products. For each finished product shall exist one primary process producing it by means of finished goods alone. The one year old machines that the primary process turns out as intermediate goods are completely used up in secondary processes with no other intermediate goods as means of production. The secondary processes produce the same finished good and possibly intermediate goods. The latter are again used in secondary processes producing the same finished good and possibly intermediate goods. The pattern repeats itself until after several stages all intermediate inputs are used up and processes are reached with the finished good as the single product.

The total output of one finished good in the economy (equal to one for $e^t B = e^t$) is thus produced by a group of one primary and several, say α , secondary processes, involving, we assume, no more than α intermediate goods which are all produced and used up within the group.*

* This last condition is optional, but must be replaced by another. See "Note" after theorem 13.2 and at the end of § 12.

This definition broadens the scheme given in Sraffa, 76 not only because it allows for varying efficiency and the use of several machines, but also, because new machines may be brought in at any stage and old machines may be used in different ways. Suppose e.g. a good G is produced by two machines (M_1, M_2) lasting two (N_1, N_2) and three years (M_1, M_2, M_3) respectively. The group will consist of four processes, if the group is to determine the price p_g of the one finished good G plus the prices $p_{M_1}, p_{M_2}, p_{M_3}$ of the three intermediate goods M_1, M_2, M_3 . Ignoring all inputs except M, N and Labour we have:

$$\text{primary process: } (1+r)(M_1 p_{M_1} + N_1 p_{N_1}) + wL_1 = G_1 p_g + M_1 p_{M_1} + N_1 p_{N_1}$$

$$\text{secondary processes: } (1+r)(M_2 p_{M_2} + N_2 p_{N_2}) + wL_2 = G_2 p_g + M_2 p_{M_2},$$

$$(1+r)\left(\frac{1}{2}M_2 p_{M_2} + \frac{1}{2}N_2 p_{N_2}\right) + wL_3 = G_3 p_g + \frac{1}{2}N_2 p_{N_2},$$

$$(1+r)\left(\frac{1}{2}M_2 p_{M_2} + \frac{1}{2}N_2 p_{N_2}\right) + wL_4 = G_4 p_g.$$

Total input of finished goods used up is $M_1 + \frac{1}{2}M_2$. The particular combination of the machines in this group makes it possible to reduce the four equations to one by multiplying them by $\frac{1}{2}[(1+r)^2 + (1+r)^2]$, $\frac{1}{2}[(1+r)^2 + (1+r)^2]$, $(1+r)$, $\frac{1}{2}(1+r)$ and adding them as in Sraffa, 76, so that intermediate goods cancel out:

$$\begin{aligned} & (1+r) \frac{1}{2} [(1+r)^2 + (1+r)^2] M_1 p_{M_1} + [(1+r)^2 + (1+r)^2] N_1 p_{N_1} \\ & + w \left[\frac{1}{2} [(1+r)^2 + (1+r)^2] L_1 + \frac{1}{2} [(1+r)^2 + (1+r)^2] L_2 + (1+r) L_3 + L_4 \right] \\ & = \left[\frac{1}{2} [(1+r)^2 + (1+r)^2] G_1 + \frac{1}{2} [(1+r)^2 + (1+r)^2] G_2 + (1+r) G_3 + G_4 \right] p_g. \end{aligned}$$

In this "pure" case, the transfer of intermediate goods between groups (interlocked systems) has been excluded.

We also exclude perennial machines, i.e. machines whose lifetime is physically indefinite (infinite), provided they are adequately maintained. The columns a_j^j, b_j^j representing a perennial machine j in a joint production system A, B are characterized by positive elements in pairs in the same line of production (e.g.

$$a_1^j = b_1^j > 0, a_2^j = b_2^j > 0, a_j^j = b_j^j = 0, 2 < j \leq n)$$

while there is one line i for which $b_i^j > 0, a_i^j = 0$ (e.g. $a_3^1 = 0, b_3^1 > 0$) so that the machine is produced somewhere (process i) and reappears on the right hand side of the equation if it has been used on the left. It cannot be said whether such perennial machines are finished or intermediate goods and they

are therefore not to be considered as an instance of pure fixed capital. They constitute the borderline between fixed capital and land, giving rise to a sort of quasi-rent. We shall deal with them in a separate section.

We shall not deal at all, however, with superimposed joint production and interlocked systems although their exclusion can at least partially be relaxed by combining the results of part one and part two of this paper.

The intuitive description of pure fixed capital systems is now replaced by a more workable mathematical definition:

Definition

A pure fixed capital system is given by a basic joint production system A, B, \mathbf{J} (A, B are (n, n) -matrices), together with an (n, n) -matrix $Q(r) \geq 0$ subdivided as follows ($m < n$):

$Q_1^1 = J$	$Q_1^2 = K$	m	A_1^1	$A_1^2 = 0$	m	B_1^1	B_1^2
$Q_2^1 = 0$	$Q_2^2 = I_{(n-m)}$		A_2^1	A_2^2		B_2^1	B_2^2
		$n-m$			$n-m$		

and with

- a) $A_1^1 \neq 0, A_1^2 = 0, B_1^1 \geq 0$ and diagonal.
- b) $Q_2^1 = 0, Q_2^2 = I_{(n-m)}$.
- c1) $Q = (q_i^j)$. If $q_i^j \neq 0, q_i^j(r)$ is a polynomial $q_i^j(r)$ with positive coefficients in $1+r$, and with $q_i^j(0) = 1$.
- c2) $Q_1^1(r) = J(r)$ is diagonal (thus $J(c) = I_{(m)}$).
- c3) $Q_1^2(r) = K(r) = [k_1^1, \dots, k_1^{n-m}]$. All elements of k_1^i except one vanish.
- c4) $Q_2^2(r) = L(r) = [l_{21}^1, \dots, l_{21}^{n-m}]$. The degree of the polynomial $j_i^1(r)$ on the diagonal of $J(r) = Q_1^1(r)$ exceed the degree of all other polynomial on the same row of Q_1 , i.e. it exceeds the degrees of all polynomials of $k_1^i(r)$.
- d) $K B_2^1$ is a diagonal matrix.
- e) $Q_1(B_2^2 - (1+r)A_2^2) = J B_2^2 + K(L_2^2 - (1+r)A_2^2) = 0$

for all r .

* f *) If k_1^1, \dots, k_1^{n-m} are the positive elements of k_1 and if Δ_1 is the $(n-m, n-m)$ diagonal matrix with the elements of k_1 on the

diagonal, the matrix

$$\begin{bmatrix} \Delta_1 & A_2^2 \\ \Delta_1 & B_2^2 \end{bmatrix}$$

has not more than α positive columns.

We translate this definition back into intuitive language:

$$A^1 = \begin{bmatrix} A_1^1 \\ A_2^1 \end{bmatrix}, \quad B^1 = \begin{bmatrix} B_1^1 \\ B_2^1 \end{bmatrix}$$

are the finished, A^2, B^2 the intermediate goods.

$$A_1 = [A_1^1, A_1^2] = [A_1^1, 0] \quad (\text{see (a)}), \quad B_1 = [B_1^1, B_1^2] \quad \text{the primary, } A_2, B_2 \text{ the secondary processes.}$$

Q is the matrix which effects the summing-up of the secondary processes and the (one) primary process (see (b), (c)) belonging to the same group (d). "(e)" means that the intermediate goods cancel out. "($\neq f \neq$)" is the optional condition which may be replaced according to the comment following on theorem 13.2. It means that if α secondary processes are in the group belonging to finished good i , at most α intermediate goods are involved.

Obviously, the determinant $\det Q(r) = \det J(r)$ does not vanish for $r \geq 1$. The matrix Q has therefore an effect similar to M in § 2,3: it generates a system where the prices of finished products are, like basics, seemingly independent of intermediate goods. But here, the coefficients are variable:

$$p(r) = w(B - (1+r)A)^{-1}\lambda = w(\tilde{B}(r) - (1+r)\tilde{A}(r))^{-1}\tilde{\lambda}(r)$$

with $\tilde{B}(r) = Q(r)B, \quad \tilde{A}(r) = Q(r)A, \quad \tilde{\lambda} = Q(r)\lambda$.

Because of "(e)" the system decomposes:

$$(J(r)B_1^1 + K(r)B_2^1) - (1+r)(J(r)A_1^1 + K(r)A_2^1)p_1 = w(J\ell_1 + K\ell_2)$$

$$(B_2^1 - (1+r)A_2^1)p_1 + (B_2^2 - (1+r)A_2^2)p_2 = w\ell_2.$$

$\tilde{B}_1^1 = J(r)B_1^1 + K(r)B_2^1$ is a diagonal matrix with non-vanishing diagonal.* We divide the equations by

* For $Q_1^1 = J(r)$, B_1^1 (a) and KB_2^1 (d) are diagonal. Since $e_{(m)}^1 \tilde{B}_1^1(c) = e_{(m)}^1 Q_1(c) B_1^1 = e_{(m)}^1 (J(c), K(c)) B_1^1 = e^1 B_1^1 = e_{(m)}^1$, it follows $\tilde{B}_1^1(c) = I_{(m)}$.

$\hat{j}_i(r) + (k_i(r) \cdot b^i)$ and denote the resulting matrices and the labour vector by:

$$\hat{A}(r) = (\tilde{B}_1^1)^T \tilde{A}_1^1 = (JB_1^1 + KA_2^1)^{-1} (JA_1^1 + KA_2^1),$$

$$I_{(m)} = (\tilde{B}_1^1)^{-1} \tilde{B}_1^1,$$

$$\hat{l} = (\tilde{B}_1^1)^T \tilde{l} = (JB_1^1 + KA_2^1)^{-1} (Jl_1 + Kl_2).$$

Thus, we get the following system determining the prices p_1 of finished products:

$$p_1 = w (I_{(m)} - (1+r) \hat{A})^{-1} \hat{l}.$$

It has the property that $\hat{l}_i, \hat{a}_i j(r)$ are rational functions in $(1+r)$ with non-negative coefficients. According to Sraffa, [76], this rational function equals $\frac{r(1+r)^{n-1}}{(1+r)^n - 1}$ for a machine with "a lifetime of n years and constant efficiency". Generalizing that result, one may interpret $\hat{a}_i j(r)$ as depreciation-quota, if j is a machine.

Note:

To conclude this chapter, we give two examples which are useful for the discussion of the axiomatic system:

I) The first example is of a complete fixed capital system fulfilling all axioms except A_7 . There are two kinds of machines, blast-furnaces M and lorries N , each growing two years old ($M_0, M_1; N_0, N_1$) and producing together steel (G) by which they are also produced. Old and new lorries are then used to produce corn (X).

$$(1+r)G_1 p_g + w l_1 = M_0 p_m,$$

$$(1+r)G_2 p_g + w l_2 = N_0 p_n,$$

$$(1+r)(M_0 p_{m_0} + N_0 p_{n_0}) + w l_3 = G_3 p_g + M_1 p_{m_1} + N_1 p_{n_1},$$

$$(1+r)(M_1 p_{m_1} + N_1 p_{n_1}) + w l_4 = G_4 p_g,$$

$$(1+r)(N_0 p_{n_0} + X_1 p_x) + w l_5 = X_2 p_x + N_1 p_{n_1},$$

$$(1+r)(N_1 p_{n_1} + X_1 p_x) + w l_6 = X_4 p_x.$$

The system is basic, yet in the 'centre' corn emerges as a self-reproducing non-basic (X) (the 'centre' is defined in § 13):

$$(1+r) G_1 p_{g_1} + w l_1 = M_1 p_{m_1}$$

$$(1+r) G_2 p_{g_2} + w l_2 = M_2 p_{m_2}$$

$$(1+r)^2 (M_1 p_{m_1} + M_2 p_{m_2}) + w ((1+r) l_3 + l_4) = ((1+r) G_3 + G_4) p_g$$

$$(1+r)^2 M_1 p_{m_1} + ((1+r)^2 X_1 + (1+r) X_3) p_x + w ((1+r) l_5 + l_6) = ((1+r) X_2 + X_4) p_x.$$

This example shows that the axiom α is independent of (a), ..., (e).

One can easily prove that each group containing α intermediate goods will consist of at most $\alpha+1$ equations, if (a), ..., (e) hold. With $\alpha=1$, the group will consist of exactly $\alpha+1$ equations.

II) The second example shows the transfer of an intermediate good from one group to another ('interlocked system'), e.g. of a race-horse (M) from racing (G) to breeding (X):

$$(1+r) M_1 p_{m_1} + w l_1 = G_1 p_{g_1} + M_1 p_{m_1}$$

$$(1+r) M_2 p_{m_2} + w l_2 = G_2 p_{g_2} + M_2 p_{m_2}$$

$$(1+r) M_3 p_{m_3} + w l_3 = X_3 p_x + M_3 p_{m_3}$$

$$(1+r) M_4 p_{m_4} + w l_4 = X_4 p_x.$$

Such transfers have been excluded, since they lead to joint production in the 'centre':

$$(1+r)^2 M_1 p_{m_1} + w \sum_{v=0}^3 (1+r)^v l_{4-v} = ((1+r)^2 G_1 + (1+r)^2 G_2) p_g + ((1+r) X_3 + X_4).$$

Genuine examples for such transfers are hard to find in industry.

13. The Centre of Fixed Capital Systems

We analyse first the system $\hat{A}(r)$, $I(m)$, $\hat{l}(r)$ * which we shall call the centre of the fixed capital system.

Theorem 13.1

For the centre $\hat{A}(r)$, $I(m)$ we have:

$$1. \quad e_{(m)}^1 (I_{(m)} - \hat{A}(0)) > 0.$$

$$2. \quad \det(I_{(m)} - \hat{A}(0)) \neq 0.$$

Proof: 1. This is nothing else than the familiar condition $e^1(B^1 - A^1) > 0$,

$$\text{for } e_{(m)}^1 Q_1(0) = e_{(m)}^1 [](0), K(0) = [e_{(m)}^1, e_{(m-n)}^1] = e^1; \tilde{B}_1^1(0) = I_{(m)}.$$

$$2. \quad 0 \neq \det Q(0) \det (R-A) = \det [Q_1, (B^1 - A^1)] \det (B_2^2 - A_2^2)$$

$$= \det (\tilde{B}_1^1 - \hat{A}_1^1) \det (B_2^2 - A_2^2)$$

$$= \det (\tilde{B}_1^1) \det (I - \hat{A}) \det (B_2^2 - A_2^2). \text{ q.e.d.}$$

* The "centre" with its variable coefficients is of course no system in the original sense but the following theorems justify the extension of the definition.

We want to prove that the centre is basic. This is not trivial. Whole columns of A_1^1 may be zero, because some finished goods are used only in conjunction with intermediate goods, e.g. "spare parts" for old machines.

Theorem 13.2

The centre $\tilde{A}(r)$, I_r is a basic system for $r \geq 0$.

Proof: (It is interesting to note that some assumptions could be reduced to allow for superimposed joint productions^{*}). Without loss of generality, we assume $r = 0$, therefore $J = J(0) = I_m$, $K = K(0)$,

$$\tilde{B}_1^1 = J \tilde{B}_1^1 + K \tilde{B}_2^1 = I_m \quad (\text{see footnote } \S 12).$$

We have to show that if $(\tilde{A}_1^1, \tilde{B}_1^1)$ is non-basic, or with the notation:

(\tilde{A}_1^1)	(\tilde{R}_1^1)	(\tilde{A}_2^1)		
(\tilde{A}_1^1)	(\tilde{R}_1^1)	$(\tilde{A}_2^1)_2$		
(\tilde{A}_1^1)	(\tilde{R}_1^1)	$(\tilde{A}_2^1)_2$	A_2^2	
(\tilde{A}_1^2)	(\tilde{R}_1^2)			
(\tilde{A}_1^2)	(\tilde{R}_1^2)			

J	K_1			m
	K_2			
	α			
	$I_{(n-m)}$			

(and ignoring trivial permutations)

that if $(\tilde{A}_1^1)^2 = (\tilde{B}_1^1)^2 = 0$, it follows that \tilde{A}, \tilde{B} is non basic. For \tilde{A}, \tilde{B} is nonbasic, A, B is also nonbasic, because $\tilde{A} = Q(0)A, \tilde{B} = Q(0)B, \det Q(0) = 1 \neq 0$ in contradiction to our assumption that pure fixed capital systems are basic.

We show that the assumptions $(\tilde{A}_1^1)^2 = (\tilde{B}_1^1)^2 = 0$ and \tilde{A}, \tilde{B} basic lead to a contradiction. We distinguish three cases according to whether (1) all, (2) some, or (3) no column of K_1 vanishes.

1) Not all the columns of K_1 vanish for if we have $K_1 = 0$, we get

$$[(\tilde{A}_1^1)_1, (\tilde{B}_1^1)_1] = [K_1 A_2^2, (\tilde{B}_1^1)_1 + K_1 (\tilde{B}_2^1)]$$

$$= [0, (\tilde{B}_1^1)_1] = [0, 0] \quad (\text{using (f)}) \text{ and} \\ \tilde{A}, \tilde{B} \text{ is non-basic.}$$

* No use is made of (c2) and (d).

2) Assume thus that the first α columns of K_1 vanish, while the remaining $\beta = n - m - \alpha > 0$ do not and assume $\alpha > 0$. From this and (e) it follows firstly that (using (e)) the last β rows of $[A_2^2, B_2^2]$ are such that the a_{j1}^2, b_{j1}^2 are zero in at least $\alpha = n - m - \beta$ columns of A_2^2 and B_2^2 , say in the first α of each, or, extending the notation,

$$(A_2^2)_2^1 = (B_2^2)_2^1 = 0. \quad \text{It follows secondly,}$$

that $(\tilde{A}_2^2)_2^1 = (\tilde{B}_2^2)_2^1 = 0$ and this entails
(using (e) again):

$$\begin{aligned} & [(\tilde{A}_2^2)_1^1, (\tilde{B}_2^2)_1^1] \\ &= [K_1^1 (A_2^2)_1^1 + K_1^2 (A_2^2)_2^1, (\tilde{B}_2^2)_1^1 + K_1^1 (\tilde{B}_2^2)_1^1 + K_1^2 (\tilde{B}_2^2)_2^1] \\ &= [0, (\tilde{B}_2^2)_1^1] \\ &= 0. \end{aligned}$$

Since all the columns of K_1^2 have a positive element and since

$$\begin{aligned} 0 &= [(\tilde{A}_1^1)_1^1, (\tilde{B}_1^1)_1^1] = [(A_1^1)_1^1, (B_1^1)_1^1] \\ &+ K_1^2 [(\tilde{A}_2^2)_1^1, (\tilde{B}_2^2)_1^1], \end{aligned}$$

we get

$$[(A_1^1)_2^1, (B_1^1)_2^1] = [(\tilde{A}_1^1)_2^1, (\tilde{B}_1^1)_2^1] = 0.$$

Thus we have, taken together:

$$[(\tilde{A}_1^1)_1^1, (\tilde{B}_1^1)_1^1] = 0, \quad [(\tilde{A}_1^1)_2^1, (\tilde{B}_1^1)_2^1] = 0$$

$$[(\tilde{A}_2^2)_1^1, (\tilde{B}_2^2)_1^1] = 0, \quad [(\tilde{A}_2^2)_2^1, (\tilde{B}_2^2)_2^1] = 0$$

This is sufficient to show \tilde{A}, \tilde{B} is non basic, if $\alpha > 0, \beta > 0$.

3) If $\omega = 0$ and no column of K_1 vanishes we can, from

$$0 = [(\tilde{A}_1^1)^2, (\tilde{B}_1^1)^2] = [(A_1^1)^2, (B_1^1)^2] + K_1[(A_2^1)^2, (B_2^1)^2]$$

conclude that

$$[(A_2^1)^2, (B_2^1)^2] = [(\tilde{A}_2^1)^2, (\tilde{B}_2^1)^2] = 0,$$

i.e. that \tilde{A}, \tilde{B} is non basic. This completes the proof.

q.e.d.

Note: The proof of this theorem makes use of the somewhat unnatural condition ($*f*$). Some condition of the sort is required to eliminate the possibility that the system A, B decomposes once the basic intermediate goods are eliminated*. Several alternatives to ($*f*$) are conceivable of which the following is obviously the simplest: On the basis of results obtained later it could be argued that the centre which appears here as an artificial construction is in fact from the economic point of view at least as important a category as the system itself. It might not only be more expedient but even more economic to assume directly that the centre is basic and to drop the assumption that A, B is basic together with ($*f*$). At any rate, it is the former hypothesis and not the latter which is essential for all of what follows.

Theorem 13.3.

A pure fixed capital system A, B has a standard ratio R and a standard commodity. The standard ratio R is a root of the equation $\det(B - (1+r)A) = 0$, the standard commodity t is given by $t^1 = q^1(B-A) > 0$ where q , the standard multipliers, are uniquely defined and positive: $q > 0$.

The proof of this theorem relies on theorem 13.4 which we prove first:

*Example I in Note to §12.

Theorem 13.4

There is a unique $\hat{q} > 0$ and a $R > 0$ which is a root of the equation $\det(\mathbf{I}_{(m)} - (1+r)\hat{\mathbf{A}}(r)) = 0$ so that $(1+r)\hat{q}\hat{\mathbf{A}}(R) = \hat{q}$ (the centre has a standard ratio and a standard commodity).

Proof: 1) $\hat{\mathbf{A}}(r)$ is a semipositive, indecomposable matrix and so is $(1+r)\hat{\mathbf{A}}(r)$ for $r \geq 0$. There is therefore for each $r > 0$ a uniquely defined $\lambda(r) > 0$ and $x(r) > 0$ with

$$\lambda(r)x(r) = (1+r)\hat{\mathbf{A}}(r)x(r)$$

and

$$e_{(m)}^1 x(r) = 1.$$

2) We have $\lambda(0) < 1$ from the Lemma in §2 and from Theorem 13.1.

3) On the other hand $\lambda(r) \rightarrow \infty$ for $r \rightarrow +\infty$.

To see this, consider

$$\lambda(r) = e_{(m)}^1 x(r) = (1+r) e_{(m)}^1 \hat{\mathbf{A}}(r)x(r).$$

$e_{(m)}^1 \hat{\mathbf{A}}(r)$ is a rowvector of rational functions in $1+r$.

$e_{(m)}^1 \hat{\mathbf{A}}^j(r)$ (which itself is a sum of rational functions) tends to a positive value (finite or infinite), if and only if there is i ($1 \leq i \leq m$, $1 \leq j \leq m$) so that $a_{ij} \neq 0$ for then and only then is the degree of the numerator with certainty at least equal to that of the denominator* for at least one of the rational functions added up to $e_{(m)}^1 \hat{\mathbf{A}}^j(r)$ (namely $\hat{\mathbf{A}}_i(r)$). If no such $a_{ij} \neq 0$ exists, the commodity j does not appear as an input in the primary processes although it is a finished good. Such a good j has therefore the character of a "spare part", being used only as an input together with old machines.

If no spare parts exist, $(1+r)e_{(m)}^1 \hat{\mathbf{A}}^j(r)$ tends to infinity for all j , thus $\lambda(r) \rightarrow \infty$ for $r \rightarrow \infty$.

*The degree is higher in the numerator if $a_{ij} > 0$, $b_i^j = 0$ for some i , given j (axiom (c4)).

But even if some spare parts exist, or more generally, if A_1^1 is decomposable ($A_1^1 \geq 0$, $A_1^1 \neq 0$ following (a)), there is (after suitable rearrangement) an indecomposable (s,s) matrix

$$(A_1^1)_1^1, \quad 0 < s \leq m, \quad \text{contained in } A_1^1.$$

$d = [e_{(1)}, 0, \dots, 0]$ is an m -vector, the first s components of which are equal to one. We use it to normalize $x(r)$:

$$d^T x(r) = 1 \quad \text{and get (with the usual notation):}$$

$$\begin{aligned} \lambda(r) &= \lambda(r) d^T x(r) = (1+r) d^T \hat{A} x(r) \\ &\geq (1+r) d^T [\hat{a}_1^1(r), \dots, \hat{a}_s^1(r)] (x_1(r), \dots, x_s(r)) \\ &= (1+r) e_{(1)}^T \hat{A}_1^1 x^1(r). \end{aligned}$$

Since $(A_1^1)_1^1 \geq 0$ and indecomposable and since $\hat{a}_i^1(r)$ tends at least to a finite positive value if $a_i^1 > 0$, $e_{(1)}^T \hat{A}_1^1(r)$ tends in each component to a positive finite value or to $+\infty$.

Thus, there is r_A , $\epsilon > 0$ so that

$$e_{(1)}^T \hat{A}_1^1(r) \geq \epsilon e_{(1)}^T$$

for $r > r_A$. From

$$\begin{aligned} \lambda(r) &\geq (1+r) e_{(1)}^T \hat{A}_1^1(r) x^1(r) \\ &\geq (1+r) \epsilon e_{(1)}^T x^1(r) = (1+r) \epsilon \end{aligned}$$

for $r > r_A$, it follows $\lambda(r) \rightarrow \infty$ for $r \rightarrow \infty$.

4) $\lambda(r)$ is a continuous function of r , $r \geq 0$, since the $\hat{a}_i^1(r)$ are continuous functions of r and since the dominant root of an indecomposable semi-positive matrix is a continuous function of the elements of the matrix.

5) Since $\lambda(0) < 1$, $\lambda(r) > 1$ for sufficiently great r and since $\lambda(r)$ is continuous, there must be a definite smallest $R > 0$ for which $\lambda(R) = 1$, so that

$$x(R) = (1+R) \hat{A}(R) x(R)$$

and

$$\det(I_{m \times m} - (1+R) \hat{A}(R)) = 0.$$

6) Since the root of $\det(I_{m \times m} - (1+r) \hat{A}(R)) = 0$ (R fixed) is simple,

$\hat{A}(R)$ being indecomposable, it follows that there is a unique $\tilde{q} > 0$ with

$$(1+R) \tilde{q}^1 \hat{A}(R) = \tilde{q}^1.$$

q.e.d.

The proof of the preceding theorem is now fairly obvious:

$\det(I_{(n)} - (1+R)\hat{A}(R)) = 0$ is equivalent to $\det(\tilde{B}_1(R) - (1+R)\tilde{A}_1(R)) = 0$.
Since R is a simple root of $\det(\tilde{B}_1(R) - (1+R)\tilde{A}_1(R))$,

there is a unique \tilde{q} so that

$$(1+R) \tilde{q}^1 \tilde{A}_1(R) = \tilde{q}^1 \tilde{B}_1(R)$$

or

$$R \tilde{q}^1 = \tilde{q}^1 \tilde{A}_1(R) (\tilde{B}_1(R) - \tilde{A}_1(R))^{-1}$$

$$= \tilde{q}^1 \tilde{A}_1(R) (I_{(n)} - \hat{A}(R))^{-1} (\tilde{B}_1(R))^{-1}.$$

$\hat{A}(R)$ is indecomposable and its dominant root is $(1+R)^{-1} < 1$. The inverse $(I(R) - \hat{A}(R))^{-1}$ does therefore exist and is positive.
Since $\tilde{A}_1(R)$, $(\tilde{B}_1(R))^{-1}$ (the latter is diagonal) are nonnegative and have (as is easily proved with 13.2) at least one positive element in every row and column,

$\tilde{A}_1(R) (I_{(n)} - \hat{A}(R))^{-1} (\tilde{B}_1(R))^{-1}$ is positive, hence indecomposable, and \tilde{q} must be positive. Define

$$q^1 = \tilde{q}^1 Q_1(R) = \tilde{q}^1 [J(R), K(R)]^*$$

and we get (using (e)):

$$\begin{aligned} (1+R) q^1 A &= (1+R) \tilde{q}^1 Q_1(R) A \\ &= [(1+R) \tilde{q}^1 \tilde{A}_1(R), \tilde{q}^1 (1+R) \tilde{A}_1^2(R)] \\ &= [\tilde{q}^1 \tilde{B}_1(R), \tilde{q}^1 \tilde{B}_1^2(R)] \\ &= \tilde{q}^1 Q_1(R) B \\ &= q^1 B. \end{aligned}$$

Obviously $q > 0$

q.e.d.

* In accordance with Sraffa, p. 84.

Theorem 13.5

The standard prices $p_{t1}(r)$ of the finished goods of a pure fixed capital system are positive for all rates of profit.

Proof: Almost obvious from $p_{t1}(r) = w(I_{(m)} - (1+r)\hat{A})^{-1}\hat{f}$. We have $\hat{f} > 0$ and $\text{dom}(1+r)\hat{A}(r) < 1^*$ for $r < R$, since R was defined as the lowest value for which $\text{dom}(1+r)\hat{A}(r) = 1$. Thus, $(I_{(m)} - (1+r)\hat{A})^{-1} > 0$ and $p_{t1}(r) > 0$ for $0 \leq r < R$.

whatever the normalization. Moreover, we have $(1+R)\hat{A}(R)x(R) = x(R)$ with $x(R) > 0$. It remains to be shown that $p_{t1}(r)$ tends to the solution of $(1+R)\hat{A}(R)x(R) = x(R)$, provided p_{t1} and x are normalized with the same vector $t^1 = q^1(I_{(m)} - \hat{A}(R))$. We omit the proof which is similar to the one given in §.

Theorem 13.6

The centre is all-engaging.

Note: By this somewhat loose formulation we mean the assertion that all activities (including the secondary processes) have to be engaged for the production of any finished good to take place (if the maximum life times of machines, i.e. the secondary processes to be employed in conjunction with the primary processes, are given).

Proof: Assume $c^1 = (c_1, 0, \dots, 0) \geq 0$ is to be produced.

$c_1 = (d_1, \dots, d_m)$ is a vector of finished goods.

(No intermediate goods shall appear in the surplus: the last $n-m$ components of the vector c vanish.) We have to show that the activity levels appropriate for the production of c^1 are positive:

$$q^1(B-A) = c^1, \quad q^1 > 0,$$

i.e.

$$q^1 = c^1(B-A)^{-1} > 0, \quad \text{if } c^1 \geq 0.$$

* $\text{dom } M$ denotes the dominant root of an indecomposable matrix M .

This follows from

$$\begin{aligned}
 q^1 &= c^1 (B - A)^{-1} = c^1 (B - A)^{-1} (Q(0))^{-1} Q(0) \\
 &= c^1 (QB - QA)^{-1} Q \\
 &= (c_1, 0) \begin{bmatrix} I - \hat{A}(0) & 0 \\ B_2^1 - A_2^1 & B_2^2 - A_2^2 \end{bmatrix}^{-1} Q \\
 &= [c_1 (I_{(m)} - \hat{A}(0))^{-1}, 0] \begin{bmatrix} J(0) & K(0) \\ 0 & I_{(n-m)} \end{bmatrix} \\
 &= [c_1 (I_{(m)} - \hat{A}(0))^{-1}](0), c_1 (I_{(m)} - \hat{A}(0))^{-1} K(0)] \\
 \text{from } (I_{(m)} - \hat{A}(0))^{-1} &> 0, c_1 \geq 0, J(0) = I_{(m)}, \\
 \text{and the fact that } K(0) &\text{ has a positive element in each column.}
 \end{aligned}$$

q.e.d.

Corollary: The centre is all-productive.

14. Pricemovements of Finished Goods

Although the centre \hat{A} , \hat{J} looks now, formally and with its positive prices, very much like a single product system, one can prove, using the methods of \hat{P} , that there is no single product system equivalent to it. Such a proof is, however, not necessary. The main difference between the behaviour of prices in an ordinary single product system and in the centre of a fixed capital system is that in the former case $\hat{P}(r)$ (prices in terms of the wage rate) rise monotonically for $0 \leq r < R$ while this is not necessarily so for the latter, despite $P_{k_1}(r) > 0$ for $0 \leq r \leq R$.

As an example consider the following simple fixed capital system:

$$\begin{aligned}
 (1+r) K_0 \hat{P}_k + l_0 &= M_0 \hat{P}_m \\
 (1+r)(K_1 \hat{P}_k + M_0 \hat{P}_m) + l_1 &= K_1^1 \hat{P}_k + M_1 \hat{P}_m \\
 (1+r)(K_2 \hat{P}_k + M_1 \hat{P}_m) + l_2 &= K_2^1 \hat{P}_k.
 \end{aligned}$$

In the first process, the new machine is produced by means of corn (k); in the second corn and a one year old machine are produced by means of corn and a new machine, in the third corn by means of an old machine.

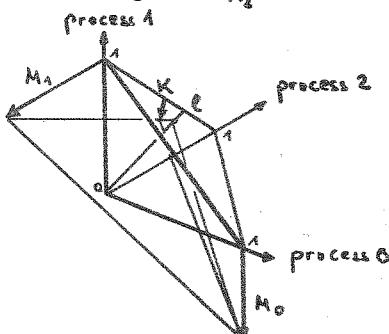
If the processes, particular the first, use little corn and if the second process produces more corn than the third while the third employs more labour than the second, and the first employs very little, it follows from $(M_0 = M_1 = 1)$

$$\hat{P}_k = \frac{(1+r)^2 l_0 + (1+r) l_1 + l_2}{(1+r) K_0' + K_2' - (1+r)((1+r)^2 K_0 + (1+r) K_1 + K_2)}$$

that $\hat{P}_k(r)$ may fall over some range of r , for in the extreme case of $K_0 = K_1 = K_2 = \epsilon, l_0 = 5; \epsilon$ small, therefore

$$\hat{P}_k \approx \frac{(1+r) l_1 + l_2}{(1+r) K_0' + K_2'},$$

\hat{P}_k falls for $\frac{l_1}{l_2} < \frac{K_0'}{K_2'}$. The diagram



shows, however, that the price of the (inefficient) one year old machine is negative in this case.

It turns out that $\hat{P}_k(r)$ will rise monotonically with r at all rates of profit where the fixed capital system is viable, i.e. where $\hat{P}_k > 0$, i.e. in particular, where the prices of all intermediate goods are positive ($\hat{P}_k > 0$).

Theorem 14.1

The prices $\hat{P}_1 = (I_{(m)} - (1+r)\hat{A}(r))^{-1} \hat{l}(r)$ are monotonically rising in a neighbourhood of $r=r_0$, if $\hat{P}_2(r_0) > 0$.

Proof: By differentiation of

$$B\hat{P}(r) = (1+r)A\hat{P}(r) + l$$

with respect to r , we get

$$B \frac{d}{dr} \hat{P}(r) = (1+r)A \frac{d}{dr} \hat{P}(r) + A\hat{P}(r),$$

$$(B - (1+r)A) \frac{d}{dr} \hat{P} = A\hat{P}.$$

We premultiply this equation by $Q_1(r)$:

$$Q_1(B - (1+r)A) \frac{d}{dr} \hat{P} = Q_1 A \hat{P}$$

and use axiom (e):

$$(\tilde{B}_1 - (1+r)\tilde{A}_1) \frac{d}{dr} \hat{P}_1 = \tilde{A}_1 \hat{P},$$

$$(I - (1+r)\hat{A}) \frac{d}{dr} \hat{P}_1 = (\tilde{B}_1)^{-1} \tilde{A}_1 \hat{P}.$$

Since $(I - (1+r)\hat{A})^{-1} > 0$, $(\tilde{B}_1)^{-1} > 0$, $\tilde{A}_1 > 0$, $\hat{P}(r) > 0$ at $r=r_0$ and since \hat{A} indecomposable, the theorem follows from

$$\frac{d}{dr} \hat{P}(r_0) = (I - (1+r_0)\hat{A})^{-1} (\tilde{B}_1)^{-1} \tilde{A}_1 \hat{P} = (I - (1+r_0)\hat{A})^{-1} [\hat{A}_1 (\tilde{B}_1)^{-1} \tilde{A}_1] \hat{P}(r_0) > 0.$$

q. e. d.

There is another sufficient condition which ensures rising $\hat{P}(r)$.

In our example, \hat{P}_k fell in a case where the old machine M_A was inefficient: though more labour was used in conjunction with M_A , less was produced than with M_0 .

In order to illustrate this concept of efficiency, we have to reinterpret our construction of the centre. As in §12, there are two machines N, M lasting two years (N_0, N_1) and three years (M_0, M_1, M_2) respectively. For a change, we combine them into a different group of course again four equations:

$$(1+r)(M_0 p_{M_0} + N_0 p_{N_0}) + w l_1 = G_1 p_g + M_1 p_{M_1} + N_1 p_{N_1},$$

$$(1+r)(M_1 p_{M_1} + N_1 p_{N_1}) + w l_2 = G_2 p_g + M_0 p_{M_0},$$

$$(1+r)(2M_1 p_{M_1} + 2N_0 p_{N_0}) + w l_3 = G_3 p_g + 2M_2 p_{M_2} + 2N_1 p_{N_1},$$

$$(1+r)(2M_2 p_{M_2} + 2N_1 p_{N_1}) + w l_4 = G_4 p_g.$$

Multiplying these equations successively by $(1+r)^2$, $(1+r)^3$, $\frac{1}{2}((1+r)^2 + (1+r))$, $\frac{1}{2}((1+r) + 1)$ and adding we get again an equation in which G appears as single product:

$$(1+r)[((1+r)^2 + (1+r)^2)M_0P_{n_0} + ((1+r)^3 + (1+r)^2 + (1+r))N_0P_{n_0}] + W[(1+r)^3\lambda_{1,1} + \frac{1}{2}\ell_1] = [(1+r)^3G_1 + \frac{1}{2}G_2]P_2$$

The mathematical operation by which all intermediate goods were eliminated to determine the price P_2 in one equation can now be made concrete as follows: instead of thinking of the four processes as of four industries running side by side at the same time, imagine an entrepreneur producing G in four successive years. He will start with buying two new machines M_0, N_0 . At the beginning of the second year he will be left with M_1, N_1 , having produced G_1 . In the second year he combines a new M_0 with N_1 and a new N_0 with M_1 and produces $\frac{1}{2}G_3, G_2, M_1, M_2, N_1$. In the third year he produces with another new N_0 and with M_1 again $\frac{1}{2}G_3, M_2, N_1$ and with M_2 and N_1 $\frac{1}{2}G_4$. Without buying any new machine he can repeat the last step with M_2, N_1 producing another $\frac{1}{2}G_4$ and nothing else.

In this way the production of four successive years appears as an integrated process using 'dated inputs' and producing a flow of one single output. The equation

$$(1+r)[((1+r)^2 + (1+r)^2)M_0P_{n_0} + ((1+r)^3 + (1+r)^2 + (1+r))N_0P_{n_0}] + W[(1+r)^3\lambda_{1,1} + \frac{1}{2}\ell_1] = [(1+r)^3G_1 + \frac{1}{2}G_2]P_2$$

represents a single product price equation for this integrated process. Each input and output is dated by the appropriate power of $1+r$.

Nothing prevents us from applying this interpretation to the coefficients of the polynomials in $1+r$ in the system $(\tilde{A}_i^j, \tilde{\beta}_i^j, \tilde{\ell}_i)$ which appear as numerators and denominators in the centre $(\hat{A}, I, \hat{\ell})$. Write

$$\hat{\alpha}_i^j(r) = \frac{\sum_{v=0}^N \alpha_{i,v}^j (1+r)^{N-v}}{\sum_{v=1}^N \beta_{i,v} (1+r)^{N-v}}, \quad \hat{\ell}_i(r) = \frac{\sum_{v=1}^N \alpha_{i,v}^0 (1+r)^{N-v}}{\sum_{v=1}^N \beta_{i,v} (1+r)^{N-v}}.$$

$\alpha_{i,j}^j, j=0, \dots, m_i$, represent then inputs of finished goods and labour to the output $\beta_{i,j}$ of good i in the first year.

$\alpha_{i,N}^j$, $\beta_{i,N}$ are inputs and output of the last year of the integrated process lasting N years (thus the N-th is the current year).

With this, we can state a sufficient condition for rising $\hat{p}_i(r)$ in the whole range $0 \leq r < R$:

Theorem 14.2

The prices $\hat{p}_i(r)$ do not fall with r for $0 \leq r < R$, if $\alpha_{i,j}^j \beta_{i,\mu} \geq \alpha_{i,j}^j \beta_{i,\nu}$, or (provided $\alpha_{i,j}^j \alpha_{i,\nu}^j \neq 0$), if

$$\frac{\beta_{i,M}}{\alpha_{i,\mu}^j} \geq \frac{\beta_{i,\nu}}{\alpha_{i,\nu}^j} \quad \text{for } j=0, \dots, m;$$

for $i=1, \dots, m$; and $\mu > \nu$.

That to say, $\hat{p}_i(r)$ rises, if the relation of output to input is less favorable at the beginning of the integrated process than at its end. If there is only one machine, it means that all the proportions of the output to each input increase as the machine grows older. Or, we may also say, $\hat{p}_i(r)$ rises, if the efficiency of the machines employed does not fall with the vintage of the machines.

Proof: (of theorem) 14.2.) Since $\hat{A}(r)$ is indecomposable, \hat{p}_i will rise, if all $\hat{Q}_{i,j}(r)$ and $\hat{I}_{i,j}(r)$ are rising functions of r . The condition of the theorem can be derived as a sufficient condition by means of a short calculation and some simplifications from the requirement $\frac{d}{dr} \hat{Q}_{i,j} \geq 0$, $\frac{d}{dr} \hat{I}_{i,j} \geq 0$.

q. e. d.

As a result, we have seen that the movements of the price of a finished good depends on the viability of the processes using the intermediate goods ($\hat{p}_i > 0$) and on the changing input-output-pattern (efficiency pattern) of the corresponding integrated process. In the following section we shall give a precise definition of rising and falling efficiency and we shall analyse the interrelations between these problems by analysing the behaviour of prices of intermediate goods. In contrast to \hat{p}_i , prices \hat{p}_i are capable of turning negative and a fortiori of fluctuating in $0 \leq r < R$.

Note: $\hat{P}_1(0)$ is the vector of the sums of direct and indirect labour ($\frac{1}{2}4$) embodied in the finished goods. If the equations in the centre are interpreted as integrated processes, the series

$$\begin{aligned}\hat{P}_1(\omega) &= (\mathbf{I} - (1+r)\hat{\mathbf{A}}(\omega))^{-1}\hat{\mathbf{l}}(\omega) = (\mathbf{I} - (1+r)(\tilde{\mathbf{C}}_1(\omega))^{-1}\tilde{\mathbf{A}}_1(\omega))^{-1}(\tilde{\mathbf{C}}_1(\omega))^{-1}\tilde{\mathbf{l}}(\omega) \\ &= (\mathbf{I} + (1+r)(\tilde{\mathbf{C}}_1)^{-1}\tilde{\mathbf{A}}_1 + (1+r)^2[(\tilde{\mathbf{C}}_1)^{-1}\tilde{\mathbf{A}}_1]^2 + \dots)(\tilde{\mathbf{C}}_1)^{-1}\tilde{\mathbf{l}}\end{aligned}$$

which converges for some rates of profit $r > 0$ is akin to a reduction to 'dated quantities of labour' ($\frac{1}{2}6$). The proper reduction to dated quantities of labour is, in accordance with Sraffa, § 79, not always possible for the whole fixed capital system. Proof: if in the example of § 14 $K_2 - K_3 = 0$ and if $K_3^l < K_2^l$, i.e. if the machine is of falling efficiency, the equation $\det(B - (1+r)\mathbf{A}) = 0$ will have a negative root which is smaller in absolute value than standard ratio R .

III. Various Types of Machines

15. The Value of Machines

To simplify matters, we consider one machine M with a total physical lifetime T at its $T - 1$ vintages M_0, \dots, M_{T-1} .

M_0, \dots, M_{T-1} produce the same finished good, say b^1 with price \hat{P}_1 . M_0 is the new machine (finished good), $\hat{P}_{m_0}, \dots, \hat{P}_{m_{T-1}}$ are the prices of M_0, \dots, M_{T-1} . We assume that there is only one machine (M_0, \dots, M_{T-1}) engaged in the production of b^1 .

The T equations

$$\begin{aligned}(1+r)a_1\hat{P}_1 + (1+r)m_0\hat{P}_{m_0} + l_1 &= b_1^1\hat{P}_1 + m_1\hat{P}_{m_1} \\ (1+r)a_2\hat{P}_2 + (1+r)m_1\hat{P}_{m_1} + l_2 &= b_2^1\hat{P}_2 + m_2\hat{P}_{m_2} \\ &\vdots \\ (1+r)a_T\hat{P}_T + (1+r)m_{T-1}\hat{P}_{m_{T-1}} + l_T &= b_T^1\hat{P}_T\end{aligned}$$

are part of the basic system A, B. \hat{P}_m , as the price of a finished good is determined in the system of finished goods - the centre $\hat{\mathbf{A}}, \hat{\mathbf{l}}$ - relative to which an old machine is a sort of non basic. We shall use the properties finished goods are known to possess to discuss the prices of intermediate goods.

With $\hat{P}_m = (\hat{P}_{m_0}, \dots, \hat{P}_{m_{T-1}})', \hat{P} = \hat{P}_1 =$

price vector of finished goods, $b_T = (b_T^1, 0, \dots, 0)$,

our equations can be written

$$\hat{P}_{m_0} - \frac{M_1}{(1+r) M_0} \hat{P}_{m_1} = [(b_1 - (1+r)a_1) \hat{p} - l_1] \frac{1}{(1+r) M_0}$$

$$\hat{P}_{m_{T-2}} - \frac{M_{T-1}}{(1+r) M_{T-2}} \hat{P}_{m_{T-1}} = [(b_{T-1} - (1+r)a_{T-1}) \hat{p} - l_{T-1}] \frac{1}{(1+r) M_{T-2}}$$

$$P_{m_{T-1}} = [(b_T - (1+r)a_T) \hat{p} - l_T] \frac{1}{(1+r) M_{T-1}}$$

or, assuming without loss of generality $M_0 = M_1 = \dots = M_{T-1} = 1$,

and with the matrix

$$N(r) = \begin{bmatrix} 1 & -\frac{1}{1+r} & 0 & \dots & 0 \\ & 1 & -\frac{1}{1+r} & \ddots & \vdots \\ 0 & & 1 & \ddots & -\frac{1}{1+r} \\ & & & \ddots & 1 \end{bmatrix}$$

and the vector

$$L(r) = \frac{1}{1+r} \begin{bmatrix} (b_1 - (1+r)a_1) \hat{p} - l_1 \\ \vdots \\ (b_T - (1+r)a_T) \hat{p} - l_T \end{bmatrix}$$

as

$$N(r) \hat{p}_m = L(r).$$

Since (proof by induction)

$$(N(r))^{-1} = \Delta(r),$$

$$\Delta(r) = \begin{bmatrix} 1 & \frac{1}{1+r} & \frac{1}{(1+r)^2} & \dots & \frac{1}{(1+r)^{T-1}} \\ & 1 & \frac{1}{1+r} & \ddots & \frac{1}{(1+r)^{T-2}} \\ 0 & & 1 & \ddots & \vdots \\ & & & \ddots & \frac{1}{1+r} \\ & & & & 1 \end{bmatrix},$$

we get finally

$$\hat{p}_m = \Delta(r) L(r).$$

Thus, we have at once:

Theorem 15.1

$$\text{If } L(r) > 0, \hat{p}_m(r) = \Delta(r) L(r) > 0.$$

This result is hardly surprising, since $(1+r) L_T(r) , 1 \leq T \leq T$ represents current net output, that is to say, the difference between the value of current output and input of the T year old machine. The machine has positive value as long as the L_T

are positive, i.e. as long as the value of current net output is positive.

We distinguish three types of efficiency, defined as constant, if $L_1(r) = L_2(r) = \dots = L_T(r)$ for a given r , rising, if $L_1(r) \leq \dots \leq L_T(r)$, falling, if $L_1(r) \geq \dots \geq L_T(r)$ for given r .

Theorem 15.2

The relations

$$b_1 = \dots = b_T, a_1 = \dots = a_T, l_1 = \dots = l_T$$

$$b_1 \leq \dots \leq b_T, a_1 \geq \dots \geq a_T, l_1 \geq \dots \geq l_T$$

$$b_1 \geq \dots \geq b_T, a_1 \leq \dots \leq a_T, l_1 \leq \dots \leq l_T$$

are sufficient for the efficiency in this sense to be constant, rising and falling respectively for all r , $0 \leq r < R$.

Proof: Obvious, since $\hat{P} = \hat{P}_1(r) > 0$ for $0 \leq r < R$.

q.e.d.

Theorem 15.3

1.

$$\frac{P_{M_T}(r)}{P_{M_0}(r)} = \frac{(1+r)^T - (1+r)^t}{(1+r)^T - 1}, \quad 0 < r < R, \quad 1 \leq t \leq T-1,$$

$$\frac{P_{M_T}(0)}{P_{M_0}(0)} = 1 - \frac{\varepsilon}{T}, \quad r=0, \quad 1 \leq t \leq T-1,$$

if $b_1 = \dots = b_T$,

$a_1 = \dots = a_T$,

$l_1 = \dots = l_T$. *

2. $\hat{P}_{M_T}(\bar{r}) > 0$, if the efficiency is constant or rising at $r = \bar{r}$.

3. If the efficiency of M is falling at $r = \bar{r}$,

$\hat{P}_{M_1}(\bar{r}), \dots, \hat{P}_{M_T}(\bar{r})$ may be negative.

$(\hat{P}_{M_{T+1}}(\bar{r}), \dots, \hat{P}_{M_{T-1}}(\bar{r})) < 0$, if $\hat{P}_{M_T}(\bar{r}) < 0$.

* This result is from Sraffa, § 83 (with diagram).

Proof: 1.

$$\frac{\hat{P}_T}{\hat{P}_{m_0}} = \frac{1 + \frac{1}{1+r} + \dots + \frac{1}{(1+r)^{T-1}}}{1 + \dots + \frac{1}{(1+r)^{T-1}}} \\ = \frac{(1+r)^T - (1+r)^T}{(1+r)^T - 1}.$$

2. $\hat{P}_{m_0}(\bar{\pi}) = L_1(\bar{\pi}) + \dots + \frac{1}{(1+r)^{T-1}} L_T(\bar{\pi})$ is positive,
 M_0 being a finished good. Thus

$$(1+\bar{\pi}) \hat{P}_{m_0}(\bar{\pi}) = (1+\bar{\pi}) L_1(\bar{\pi}) + L_2(\bar{\pi}) + \dots + \frac{1}{(1+r)^{T-2}} L_T(\bar{\pi}) \\ = (1+\bar{\pi}) L_1(\bar{\pi}) + p_{m_0} > 0.$$

If $L_1(\bar{\pi})$ is positive,

$\hat{P}_{m_0}(\bar{\pi})$ is positive, because $L_T(\bar{\pi}) \geq L_1(\bar{\pi}) > 0, T \geq 2$.

If $L_1(\bar{\pi})$ is negative or zero, $\hat{P}_{m_0}(\bar{\pi}) \geq (1+\bar{\pi}) \hat{P}_{m_0}(\bar{\pi}) > 0$.

3. The reader may construct an example using the graphic technique. The rest follows, using a similar argument as in (2).

Note 1: Efficiency, as defined in this paragraph, is dependent on the rate of profit and the standard of prices. While the former is an inherent problem, the latter is not, or to a lesser extent: if all prices are positive in a system in any one standard, they are positive in all standards.

Note 2: $\frac{M_T \hat{P}_T}{L_{T+1}}$ is the capital-net output ratio for machine M_T .

For machines with constant efficiency at all rates of profit

the "capital-net-output ratio" is

$$K_T(r) = \frac{M_T \hat{P}_T}{L_{T+1}} = 1 + \frac{1}{1+r} + \dots + \frac{1}{(1+r)^{T-T-1}}, \\ 0 \leq T \leq T-1.$$

(The machine has a lifetime of T years). K_T falls monotonically with increasing r and age. Falling efficiency lowers, rising efficiency raises the capital net output ratio with respect to K_T .

The average capital-net output ratio for a machine of constant efficiency is:

$$\bar{K}_T(r) = \frac{1}{T} \sum_{T=0}^{T-1} K_T(r).$$

We get $\bar{K}_T(0) = \frac{T+1}{2}$.

16. Efficiency and Obsolescence

The results of §15 require some comment. While it is perfectly obvious that ageing machines retain certain positive values as long as they work profitably ($p_m > 0$ if $L(r) > 0$), it is more difficult to accept the results of the last theorem with the startling asymmetry between rising and falling efficiency. Prices of machines with rising efficiency are positive under all circumstances, even if the new machines seem not to work profitably (e.g. $L_1(r) < 0$, $L_2(r) < 0$ etc.); but machines of falling efficiency may be worthless, or even of negative value, from a certain age onwards. Given the expectation of an average profit, that is to say, given r , the latter have to be subsidized, if they are to be kept running up to the end of their physical lifetimes.

An intuitive explanation would have to run roughly as follows: prices of joint products can in no way be explained as cost-of-production prices. While the value of the whole product of an industry is equal to the "cost of production", this value is embodied in the various products according to the uses to which they are put as means of production. In the particular case of fixed capital, the value of a machine depends essentially on the return obtained in its later use. According to the traditional definition, the value of an asset is equal to the value of the correctly anticipated and properly discounted net returns. To calculate the return L_τ in year τ , one has to deduct current costs A_τ from current revenue B_τ . If the discount factor is $1+r$ one gets as the value $p_{m_{t-1}}$ of a $t-1$ year old machine M with lifetime T,

$$p_{m_{t-1}} = \sum_{\tau=1}^{T-t} \frac{1}{(1+r)^{\tau-1}} (B_\tau - A_\tau) = \sum_{\tau=1}^{T-t} \frac{L_\tau}{(1+r)^{\tau-1}}, \quad t=1, \dots, T.$$

*Not even the prices of basics in single product systems can be explained by cost of production alone. The term can be applied only by abstraction from the interdependence of markets, i.e. by considering the good in question as a non-basic whose price is unequivocally equal to its cost of production (see Sraffa § 9)

This formula is identical with the one above:

$$\hat{P}_m = \Delta(\cdot) L(\cdot),$$

if one bears in mind that current costs A_T are equal to $\alpha_T \hat{P}(\cdot) + b$ and revenue to $\frac{1}{\alpha_T} L(\cdot) \hat{P}_m(\cdot)$. We have to conclude that the traditional method of evaluating capital goods is a special case of the "joint production" approach.*

The puzzle of rising and falling efficiency is now easily understood: to the case of rising efficiency corresponds "learning by doing", "work in progress", etc. The extreme case is a plant whose construction takes several years during the first of which no product is obtained. If efficiency does not fall, as assumed in theorem 15.3.2, nothing could be gained by shortening the economic life of the plant. Since there is no alternative to the plant in that its product cannot be produced by any better means, a positive value is assigned even to the plant under construction.

The case of falling efficiency is quite different: prices may turn negative (economic obsolescence may occur before physical obsolescence is reached), because the old machines are in fact not indispensable. We have seen earlier that negative prices can occur only if some processes in the system can be discarded without the system becoming totally unproductive (see §8). This is indeed the case if efficiency falls: total net output of the system can be kept constant by increasing the number of new machines and shortening the average lifetime.

By reducing the lifetime of inefficient machines we arrive eventually at a system with positive prices (provided the fundamental

* Note here in particular that P_m , is, on the one hand, equal to the 'price of production' in the centre and on the other to 'discounted net returns'. This equality, although most natural from the economic point of view, had to be proved. The 'price of production' in the centre explains the price of the product from the point of view of past costs (that is the meaning of the 'integrated process' in §14). Here, in §16, we explain the price of a machine from the point of view of the future income it generates.

condition $e^t(R-A) \geq 0$ holds for some reduced system, if it does not, the system is not viable under any circumstances).

The efficiency of many machines will partly rise and partly fall. The efficiency pattern may be seen as the super-imposition of rising and falling trends.

A machine will hardly rise for ever in efficiency. On the other hand, a machine of rising efficiency is not likely to suffer a sudden death. Thus our definition of rising efficiency is only an analytical concept. Machines of rising efficiency may be supposed to turn, after a finite time into machines of constant or falling efficiency.

Perennial constant efficiency will be dealt with later.

As regards falling efficiency, the maximum lifetime is essentially determined by economic considerations. Here too, the idea of a definite age of physical obsolescence is a simplification. It may be defined as the maximum age for which $e^t(R-A) \geq 0$, that is, the maximum age compatible with the production of a surplus (this is not independent of other machines). Apart from this obsolescence has to be determined by economic considerations and is subject to the fluctuations of the rate of profit.

17. Fixed Capital, Switching of Techniques and the von Neumann Model

Here, as elsewhere, fixed capital is seen to lie in between the simple features of single product industry systems and the intricacies of joint production.

Let us first restate the argument about "switching" in basic single product systems (it can immediately be generalized to all-engaging systems).

Theorem 17.1

If we are given an all-engaging system with an alternative method of production for one industry, it follows for any given rate of profit, r , smaller than both maximum rates of profit:

i. All prices \neq (prices in terms of the wage-rate) are lower

^a: Nonsubstitution Theorem'

for one of the two techniques (the "superior") - or else they are all equal.³⁾

2. Wage-rate and real wage are higher for the superior technique at the given rate of profit.

3. The inferior technique, if it is at all capable of reaching this highest real wage, reaches it at a lower rate of profit.

We prove this theorem as follows:

1) (A, B, ℓ) denotes our original system. The question is, whether it is advantageous to replace an industry, say the first (a_1, b_1, ℓ_1) by (a_0, b_0, ℓ_0) (it is assumed, of course, that this replacement is technically feasible). Write

$$C = \begin{bmatrix} b_2 - (1+r)a_2 \\ b_n - (1+r)a_n \end{bmatrix}, \quad c_0 = b_0 - (1+r)a_0, \\ c_1 = b_1 - (1+r)a_1,$$

$$c_0 = \begin{bmatrix} c_0 \\ C \end{bmatrix}, \quad m = (l_2, \dots, l_n)', \\ m^0 = (l_0, m)',$$

$$c_1 = \begin{bmatrix} c_1 \\ C \end{bmatrix}, \quad m^1 = (l_1, m)',$$

and let \hat{p}_0, \hat{p}_1 denote the two possible price-vectors in terms of the wage rate so that

$$C_0 \hat{p}_0 = m^0, \quad C_1 \hat{p}_1 = m^1 \\ \text{Thus } C \hat{p}_0 = C \hat{p}_1 = m, \quad C(\hat{p}_1 - \hat{p}_0) = 0.$$

Consider the set of all vectors x for which $Cx = 0$.

Since $r k C = n-1$, it is a straight line g through the origin. In particular, the x_0 for which

$$C_0 x_0 = \begin{bmatrix} c_0 x_0 \\ C x_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = e^1$$

is on this line. Since $(C_0)^{-1} > 0$ (we are in an all-engaging system), $x_0 = (C_0)^{-1} e^1 > 0$ and all points on g are positive, negative or zero³⁾.

³⁾Inequality occurs at most at n rates of profit: the "switch points".

³⁾ $\hat{p}_0 = \hat{p}_1$ (switchpoint), if and only if $\ell_0 = c_0 \hat{p}_0 = c_1 \hat{p}_1 = c_0 (C_0)^{-1} m^1$, therefore at most at n points.

Thus we have either $\hat{P}_a > \hat{P}_0$, $\hat{P}_a = \hat{P}_0$ ("switch point")
or $\hat{P}_a < \hat{P}_0$.

2) Follows immediately from (1)

3) If a is the standard in which prices are measured,

$$w_a = \frac{1}{a'(\bar{B} - (1+r)A)^{1/l}} = \frac{1}{a' \hat{P}(r)}$$

is the wage rate corresponding to this standard. It falls monotonically with r , since $\hat{P}(r)$ rises monotonically with r and it is a straight line $w = A - \frac{C}{R}$, if $a' = q'(\bar{B} - A)$ (standard commodity). w_a is smaller for the inferior technique at the rate of profit at which comparison is made and w_a falls for higher rates of profit. This completes the proof.

q.e.d.

This theorem is important because it shows that once the distribution between profits and wages is given - be it specified by means of r , w or whatever - the technique which could be adopted under conditions of perfect competition because prices are lowest * is also the one most advantageous for the capitalist class as a whole (given the wage, profits are highest) and for the class of wage-earners, (given r , wages are highest).

The choice between technically alternative systems for which the assertions of theorem 17.1 hold will be called neutral.

The case of fixed capital systems is complex, because not the whole system, but only the centre fulfills the condition

$(\bar{I}_{(m)} - (1+r)\bar{A}_{(m)})^{1/l} > 0$. With this, the first and second parts of theorem 17.2 hold, if none other than finished goods are taken into account and if the methods using intermediate goods are only considered to the extent that they influence the production of finished goods.

* Warning: \hat{P} is lower for the superior technique, but not all prices need to be lower for the superior technique in some other standard.

Since wage-goods are, as a matter of course, finished products, the third part of 17.1 holds for fixed capital systems in those ranges of the rate of profit where prices of finished goods in terms of the wage rate rise monotonically with the rate of profit, i.e. in those ranges where the system is viable because all prices are positive.^{**}

Theorem 17.2

If prices $\hat{P}_1(r) > 0$ for $r_1 \leq r \leq r_2$, prices \hat{P}_A of finished goods arise monotonically with r , and the choice of techniques is neutral in the centre of the fixed capital system for $r_1 \leq r \leq r_2$.

This result is not quite as concise as the one obtained for single product systems, because the superior technique entails lower prices \hat{P} only for finished products, not necessarily for intermediate goods (that they are positive should be assumed in any case before-hand). It is as a matter of fact quite easy to construct examples where the price \hat{P}_{M_T} of an intermediate good M_T rises in consequence of an "invention" which lowers the prices of finished goods.

The question raised here, namely how a technique will come into use without all the prices being lower for the superior technique, is somewhat outside the framework of Mr. Sraffa's system.* It does not affect the fact that the technique we call superior is advantageous for both classes and that is the important criterion. Moreover, even in single-product systems it cannot be said that the superior technique has lower prices in any standard: this is warranted only for prices in terms of the wage-rate. We are therefore referred to market-prices rather than to "natural prices" with which "Production of Commodities by Means of Commodities" alone is concerned.

It is nevertheless intuitively an important argument for the superiority of a technique under conditions of perfect competition that it entails lower prices in terms of the wage-rate. It would

* Compare Sraffa [96], footnote.

** Theorem 14.1

play a part in a theory of actual prices. In order to extend such a theory to cover fixed capital, one would have to take the qualitative difference between finished and intermediate goods into account. It is after all not only a feature of our mathematical construction that the determination of prices of finished products precedes the determination of prices of intermediate goods. Prices of finished goods are determined in an actual market and the value of a finished good is realized in money, while the prices of old machines reflect estimates of expected returns the accuracy of which may only be tested upon the failure of an enterprise. Thus we propose as a provisional solution that the choice of techniques is governed by the prices of finished goods.*

If this is accepted, fixed capital must be interpreted as a case in between single product industries, where prices are - apart from their interdependence in the basic system - cost-of-production prices and land which, not being produced, has no value as such. Its price has to be derived from rent which in turn is only a residual after the values of circulating and fixed capital, wages and profits have been deducted from revenue.**

* In particular, this criterion will enable us to determine the optimal life-time for all machines in the system for given r , for which 17.2 we have proved that for all possible combinations of life-times of machines in a given system fulfilling the basic condition $e^t(B-A) > 0$ and $\hat{P} > 0$ ($\hat{P} > 0$) there exists a technique whose prices \hat{P}_i are lower than the ones of all other combinations. Since the centre is all-productive, this is independent of the proportions in which finished goods are produced. ***

** The character of fixed capital is further expressed in the peculiar way the rate of profit is equalised on old machinery, namely by determining the price of the machine in such a way that the machine fits into a system in which the rate of profit is already given. (Besides, the condition $A(B-A)^{-1} > 0$ is not fulfilled for fixed capital. (see footnote §8.))

*** If several machines are engaged in the production of one finished good, it will also be possible to determine the best of several possible combinations of the ageing machines into a group as described in §12 and §15.

We end this section with a remark on the von Neumann model. We have proved that there exists a maximum rate of profit with which a positive vector q is associated so that

$$q' (B - (1+R)A) = 0$$

(A, B is a pure fixed capital system).

Assume $p_t(R) > 0$. We have then

$$(B - (1+R)A) p = 0$$

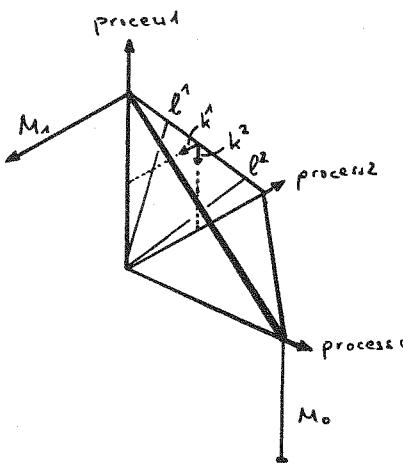
and q and p may be interpreted as von Neumann activity and price-vectors, if $1+R$ is the maximum rate of growth in the economy. But what, if $p_t(R) \leq 0$? We have seen that this is possible only if the efficiency of the machines involved does not rise with age (or remains constant). We know on the other hand that unprofitable activities are not used in the von Neumann model.

Since a von Neumann solution must exist for a basic system A, B ($A \geq 0, B \geq 0$) we conclude that there must be some unprofitable activities and that the unprofitability is due to inefficient machines. The condition that the von Neumann expansion factor is equal to the maximum possible rate of growth ensures that new machines are used in greater proportion than the inefficient old machines in such a way that the von Neumann q gives the proportions of the fastest balanced growth possible. Intermediate goods still produced, but not used, in a von Neumann activity will appear as overproduced goods and their von Neumann prices will be zero.

Thus we find that overproduced goods in a basic system are not only waste products, or miraculously abundant commodities, but old machines whose continued use would impede the growth of the system.

The inefficiency expressed in a negative labour value (see §4) is now also easily illustrated: an old machine, M_T , cannot be of rising efficiency, if it has a negative labour value. It will therefore be possible to keep the net output of the system (apart from M_T) constant by using the processes employing the new machine to a greater extent and by reducing the use of M_T . This

will save labour. At the same time, M_T will not be used up any more - hence the appearance of M_T as a product in the output of the system, produced by a negative amount of labour. In the example (for the explanation, see §14) prices are positive for small rates of



profit and negative afterwards, if corn inputs and outputs are given by k^1, k^2 . They are negative first and positive up to R, if corn is represented by k^2 , labour by l^2 .

In the first of these two cases, the process employing M_1 would be abolished, if $r \neq R$ and would thus not enter the von Neumann system; in the second it would be abolished at $r = 0$.

18. Perennial Machines

A machine M with constant efficiency^{*} and a lifetime of T years enters the centre with the production equation

$$M_0 p_{m_0} \frac{r(1+r)^T}{(1+r)^T - 1} + (1+r)a_i p + wl_i = b_i^j p_j$$

(see Sraffa §75/76). The expression $\frac{r(1+r)^T}{(1+r)^T - 1}$ tends to

r as T tends to infinity and we get for a capital good with a very long lifetime at constant efficiency:

$$r M_0 \hat{p}_{m_0} + (1+r)a_i p + wl_i = b_i^j p_j.$$

The equation can also be written as

$$(1+r)(M_0 p_{m_0} + a_i p) + wl_i = b_i^j p_j + M_0 p_{m_0}.$$

In this form, the equation is equally reasonable, since it represents the equation for a machine which is perfectly maintained

*(with constant input and output coefficients)

so that it leaves the production process in the same condition in which it entered it, i.e. undeteriorated. In consequence, the price of the machine in use is the same as the price of the new machine. This second equation is more natural from the point of view of joint production and is valid also for $r = 0$ but the first expresses a perhaps more familiar economic approach, for we can interpret

$r f_{m_0} = g_a$ for $r \neq 0$ as hire-price^x and the equation reads as follows:

$$g_a M_0 + (1+r) a_i p + w l_i = b_i^j p_j.$$

It is an economic and not a technical consideration, whether a particular machine is used as a permanent capital good (which may involve a permanent, high level of maintenance cost, provided it is at all feasible) or as machine with a finite life-time. A railwayline is genuinely everlasting, because the rails are periodically replaced. But a knife may also last forever (and this is no paradox because we are speaking of the knife's economic identity), if the handle is replaced every six and the blade every two months.^{xx} Whether these constant repairs are worthwhile, or whether it is advantageous to rely on periodic replacement of the entire machine and the extent to which the two are to be combined, are matters of choice of techniques.

Although permanent goods cannot be termed as either finished or intermediate goods, they fit into a slightly generalized concept of fixed capital systems - not surprisingly, since they are the limit case for machines of finite age and constant efficiency. The reader will verify that all propositions proved about the centre still hold^{xxx}

^x Hire-price, not quasi-rent. The latter term should be reserved for "machines of an obsolete type", still in use, but not currently produced (Sraffa, § 91). Mathematically, quasi-rents are indistinguishable from rents.

^{xx} The over-all period of production is assumed to be one year.

^{xxx} With the exception that the centre may not be "basic" at $r = 0$. Moreover, it should be assumed that some finished goods exist.

if expressions representing the use of permanent goods are included in $\hat{A}(r)$ in the form $\frac{r}{\alpha + r} \cdot M$. In particular, prices of commodities produced by means of permanent goods rise monotonically with r in terms of the wage rate, if all other prices do the same.*

*To see this, one has to start the proof of 14.1 from an equation

$$B \hat{p}(r) = A \hat{f}(r) + r C \hat{p}(r) + l$$

where A represents the matrix of all inputs of finished and intermediate goods, where C is equal to A augmented by perennial machines, and where B is the matrix of all outputs.

III Land and Rent

19. "Extensive" and "Intensive" Diminishing Returns

Unproduced means of production which, by "being in short supply enable their owners to obtain a rent"^{*} can in most cases be described as goods which leave every process in exactly the same condition as they enter it and which are nowhere produced. They present far greater difficulties than either pure consumption goods which are also non-basics but produced and not used in production or permanent capital goods which also leave the production process as they enter it but are produced.

Good k is an unproduced good or means of production ("land" for short), if $a^k = b^k = (\lambda_1, \dots, \lambda_n)^T$. A typical equation reads (λ denotes "land"):

$$(1+r)(a_i p + \lambda_i p_\lambda) + w l_i = b_i p_j + \lambda_i p_\lambda$$

(a_i is a vector of various inputs, b_i the crop produced.). It is more usual and more correct^{**} to write this equation as Mr.Sraffa does

$$(1+r)a_i p + \lambda_i p_\lambda + w l_i = b_i p_j$$

with $p_\lambda = r p_\lambda$ as the rent accruing to the owner of the land. Rent exists whether land is traded or not. If it is, its price is determined by capitalisation of the rent at the current rate of

^{*} Sraffa, § 85.

^{**} More correct, because the second form does not presuppose that land leaves the process unchanged. Mathematically however they are equivalent.

^{*****} In the case of land, $\det(B-A) = 0$, if the first form is chosen. $B^{-1}A$ (if B^{-1} exists) has then an eigenvalue $\lambda = 1$, thus the reduction to dated quantities of labour is impossible, before land is eliminated (see § 6). If the second form is chosen, $\det B = 0$.

interest and this price may be identified with the price entering the equations in the first form, if we assume, for simplicity's sake, that the rate of profit and the rate of interest are the same.

Whichever expression is chosen: it is clear that rent and price of land cannot be determined before other prices are known. Land is a non-basic, even if its product enters the basic system.*

I) Differential Rent of the First Kind (IRI):

There is only one product, grown on m different sorts of land of which one, the first, is not short in supply. Then we get m equations:

$$(1+r) a_1 p + w l_1 = b_1^j p_j$$

$$(1+r) a_2 p + \lambda_2 f_2 + w l_2 = b_2^j p_j$$

$$(1+r) a_m p + \lambda_m f_m + w l_m = b_m^j p_j.$$

The first of these equations determines p_j , the others determine the $m - 1$ rents f_2, \dots, f_m . In order to understand why one particular land is not in short supply, assume tentatively that none of the lands is in short supply; while no selection of $m - 1$ lands is sufficient to produce total output required. The assumption leads immediately to a nonsensical conclusion, for rents would still occur as an expression of surplus profits obtained on superior land at the given rate of profit, so that part of rent-yielding land would be cultivated and part not. This is impossible whether land is owned by landlords or by the producers and hence the fact that one and only one of the cultivated lands will appear not to be in short supply - the others either being fully cultivated or not cultivated at all.

Since rents depend on the rate of profit, it is thus not only not feasible to order the sorts of land according to productivity before the distribution between profits and wages is known, but even impossible to tell before hand which land will not be in short supply.

*Our theory of the basic system (1-3) was based on $\det(B-A) = 0$, $e^t B = e^t$. In order to apply this theory to land, one has to write the equations in the first form (so that $e^t B = e^t$ is a possible normalization), one has to replace A by $\bar{A} = (1+r)A$ where $\det(B-(1+r)A) \neq 0$, $(\det(B-(1+r)A) \neq 0)$, and one has to transform the rate of profit accordingly:

$$1+r \rightarrow \frac{1+r}{1+\bar{r}}$$

Production is expanded by cultivating ever new types of land and this expansion defines an order of fertility (dependent on the rate of profit). Output rises continuously, while the new areas come gradually under cultivation. Whenever a new area has been covered fully, prices and rents rise (barring a negative feedback on costs of production).

As an example outside agriculture for this type of rent (DRI) we mention the surplus profits obtained on a patented cheap method of production which allows a restricted group of entrepreneurs to produce at lower costs than all other manufacturers in the same business during the period for which the patent is granted and on the assumption that the privileged entrepreneurs are unable or unwilling to supply the whole of the market. The patent derives a definite value from the capitalization of the surplus profit.

II) Differential Rent of the Second Kind (DRII)

The situation alters if the output required makes full use of all the land available for cultivation necessary. Assume land is of only one type. Two techniques

$$(1+r)a_1 + w\ell_1 + \lambda_1 g = b_1^j p_j$$

$$(1+r)a_2 + w\ell_2 + \lambda_2 g = b_2^j p_j$$

determine rent g and price p_j . Assume for simplicity good j is non-basic and does not enter its own production. Write $p_j = r$

$$k_i = (1+r)a_i + w\ell_i$$

for the unit-cost of production and normalize $b_1^j = b_2^j = \frac{1}{2}$

to get

$$k_1 + \lambda_1 g = \frac{1}{2}$$

$$k_2 + \lambda_2 g = \frac{1}{2}.$$

Rent and price are positive, if and only if $k_i < k_j$ implies

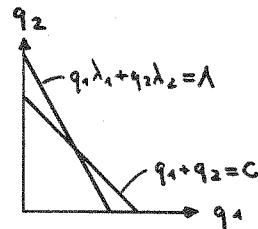
$$\lambda_j < \lambda_i, \text{ since } g = \frac{k_1 - k_2}{\lambda_2 - \lambda_1}.$$

The two techniques are compatible if one is relatively more capital-intensive, the other more land-intensive (given r).^{*}
 Assuming constant returns to scale, we determine to what extent the two techniques are to be used, if total output is given.^{**}

Let total output be C and total area of land Λ . With activity levels q_1, q_2 we must have

$$q_1 \lambda_1 + q_2 \lambda_2 = \Lambda$$

$$q_1 + q_2 = C.$$

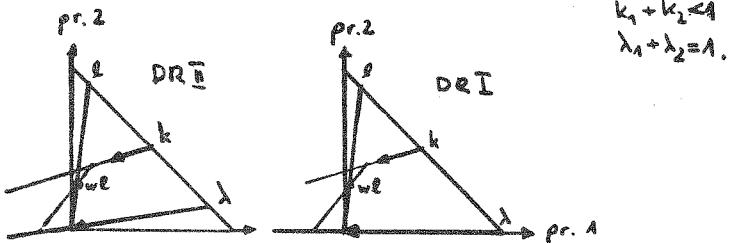


As C rises the more land-intensive technique (the technique with the smaller λ_i) covers a greater and greater percentage of the total surface. Rent and price change only when the rise in output enforces the introduction of a third yet more land-intensive technique superseding one of the two others.

^{*} For a closed model, the price-behaviour can be studied by means of a diagram:

$$(A+r)(k_1 p + \lambda_1 \pi) + w l_1 = \frac{1}{2} p + \lambda_1 \pi$$

$$(A+r)(k_2 p + \lambda_2 \pi) + w l_2 = \frac{1}{2} p + \lambda_2 \pi, \quad \pi = g, \quad l_1 + l_2 = 1,$$



For IRI, we have $\lambda_1=0, \lambda_2=1$.

^{**} See Sraffa, § 88 and / 5 /.

^{**} Land-intensity, but not capital-intensity, has here a meaning independent of the rate of profit.

In this case (IRII), we mention two different types of housing in a geographically restricted urban area as examples outside agriculture. Ground rents are reflected in land values, while the two techniques find a visible expression in the height of houses.

Bearing the two basic patterns of IRI and IRII in mind we consider heuristically more general cases; excluding multiple product processes (joint production) - apart from land itself, of course.

If, with agriculture, mining, forestry, etc, there are k types of products to be grown on m types of land, they will allow up to $m + k$ different processes* (assuming other prices as given).

If $(m, k) \gg (2, 2)$, it is not necessary for any product to be grown on any one land by two different methods, even if all land is in short supply. If there are e.g. two crops to be grown on two types of land both in short supply, it may be that both products will be grown on both lands so that both rents will be positive. If there are three products on two lands, one land will always appear to be unsuited for at least one of the crops. Except for flukes, no two crops can be grown together on more than two lands; we would have six equations for five unknowns, if $m = 3$, $k = 2$. The strict requirement of a uniform rate of profit and uniform rents thus enforces a certain specialisation of the land (with single product processes).

To illustrate this, assume all k products are grown on a no-rent land (IRI). Each rent-paying land will then in general allow only one crop (exceptions indicate "switchpoints" and do not even allow small variations in the rate of profit). If all land is in short supply (IRII) (rent is to be paid everywhere) and if all products are grown on one of them, two products will be compatible

* There are k prices and m rents to be determined which allows only $m + k$ equations (except at switchpoints).

on one other land, while all the remaining lands will be completely specialised.

This specialisation is more natural (for single product processes) than it seems: e.g. the value of a partly exploited coal-field is hardly influenced by the value of the forest which still grows over the unexploited parts.

Such specialisation does naturally not take place in the same way, if the crops are joint products, linked by technical production conditions, such as rotating crops. (Rotating crops are joint products despite the fact the plants do not grow together in the same place at the same time, because the alternation increases the fertility of the soil and makes the production of the crops concerned technically interdependent.) One could still elaborate a theory of enforced specialisation by classifying joint products into partly overlapping groups, but we shall not pursue the matter further.

20. Effects of the presence of Land on the Basic System

We now turn to the influence of the presence of land on the basic system. The general case has been dealt with in Part One. In this paragraph we discuss only single product processes using land, i.e. essentially DRI and DRII.

There is a sharp contrast in the way DRI and DRII affect the basic system. If a basic good is grown on a no-rent land (DRI), there is no problem, because it will enter the basic system represented by a single product process. If all land is in short supply however, negative multipliers will inevitably come into play, as is shown by the following example of two goods grown in single product processes on two lands both in short supply:

Corn	Cattle	Land I	Land II	Labour	Corn	Cattle
$(\frac{1}{a_1}, \frac{a_2^1}{a_1}, 1, 0, l_1)$	$\rightarrow (\frac{b_1^1}{b_1}, 0)$					
$(\frac{1}{a_2}, \frac{a_2^2}{a_2}, 1, 0, l_2)$	$\rightarrow (0, \frac{b_2^2}{b_2})$					
$(\frac{1}{a_3}, \frac{a_3^2}{a_3}, 0, 1, l_3)$	$\rightarrow (\frac{b_3^1}{b_3}, 0)$					
$(\frac{1}{a_4}, \frac{a_4^2}{a_4}, 0, 1, l_4)$	$\rightarrow (0, \frac{b_4^2}{b_4})$					

(For convenience the coefficients of land are taken to be equal to one).

The basic system involves not only negative multipliers - it leads to a sort of joint production system with some negative elements:

$$(a_1^1 - a_2^1, a_1^2 - a_2^2, l_1 - l_2) \rightarrow (b_1^1, -b_2^2)$$

$$(a_3^1 - a_4^1, a_3^2 - a_4^2, l_3 - l_4) \rightarrow (b_3^1, -b_4^2).$$

The conditions for rent and prices to be positive in certain ranges of the rate of profit and for the standard commodity to exist are very complicated. Our results derived for general joint production systems are not applicable, because of the negative coefficients.

A three sector model is sufficient to illustrate that although the goods which enter the basic system are unique the basic processes are determined only up to linear combinations of processes (compare § 3 above).

Corn	Cattle	Land	Corn	Cattle
$(\frac{1}{a_1}, \frac{a_2^1}{a_1}, 1)$	$\rightarrow (\frac{b_1^1}{b_1}, 0)$			
$(\frac{1}{a_2}, \frac{a_2^2}{a_2}, 1)$	$\rightarrow (0, \frac{b_2^2}{b_2})$			
$(\frac{1}{a_3}, \frac{a_3^2}{a_3}, 1)$	$\rightarrow (\frac{b_3^1}{b_3}, 0)$			

The basic system can be formed by subtracting one of the three equations from the two others. The resulting three systems will look different but give of course the same prices for all r.

Even if the basic system is a single product system in all respects except in that it shows one negative coefficient, all sorts of difficulties may arise. If in the example

Ploughs	Corn	Land	Labour	Ploughs	Corn
(m_1 , , k_1 , , 0, , l_1)	—→ (1, , 0)				
(m_2 , , k_2 , , λ_2 , , l_2)	—→ (0, , 1)				
(m_3 , , k_3 , , λ_3 , , l_3)	—→ (0, , 1)				

the particular values

Ploughs	Corn	Land	Ploughs	Corn
($\frac{2}{5}$, , 1, , 0)	—→ (1, , 0)			
($\frac{1}{10}$, , $\frac{2}{5}$, , 1)	—→ (0, , 1)			
($\frac{2}{5}$, , $\frac{1}{5}$, , 2)	—→ (0, , 1)			

are taken, no standard commodity exists, for the standard system (the second rows are doubled which does not alter price relations)

$$(\frac{2}{5}, 1) \rightarrow (1, 0)$$

$$(-\frac{1}{5}, \frac{3}{5}) \rightarrow (0, 1)$$

has a characteristic equation

$$\frac{41}{25} (1+r)^2 - (1+r) + 1 = 0$$

with negative discriminant

$$1 - \frac{44}{25} = -\frac{13}{25}$$

and thus no real, let alone positive solution for $1+r$ or r .

The example is striking, since neither of the two corn producing processes seems at first sight distinctly superior to the other. The second process employs less ploughs and is more land intensive, while the corn rate of reproduction is higher for process three. One discovers, however, on closer examination that process three requires four times more ploughs and since plough production has a very great input of corn one expects that l_2 must be considerably greater than both l_1 and l_3 , if the two processes are to be compatible.

One finds that both prices and $\frac{p}{r}$ are positive for $r = 4\%$, if $l_1 : l_2 : l_3 = 2 : 25 : 1$. The proportion of the labour inputs cannot be altered much, lest the rent turn negative.

The non existence of the standard commodity in this case is as such no important result; for the standard commodity is merely an analytical tool which facilitates and illuminates the explanation of the properties of self-reproducing systems. It serves its purpose particularly well in single product and related systems, like fixed capital systems where the relation

$$A = \frac{r}{q} + w_t$$

has an unambiguous meaning, because the wage falls with a rise in the rate of profit, whatever the wage goods. But as soon as prices of wage goods in terms of the wage-rate fluctuate, it is not certain any more that the wage will fall with a rise of r . Nor is it certain any more that one of two alternative techniques will be advantageous for both classes.

The possibility of the lack of these two properties is more important than the possibility of the lack of the standard commodity in systems where differential rent of the second kind occurs.*

It comes perhaps less as a surprise if it is remembered that the laws of income distribution are here (DRII) complicated by the presence of a third class of income receivers (landlords, land-owning capitalists, etc.)

The existence of a third form of revenue (rent) will not invariably destroy the simple laws of income distribution which govern single product industries. We have seen this in the case of differential rent of the first kind with single product processes. If all land is in short supply, the problem need not occur either. Whether it will, depends for instance on whether or not the basic system from which rent is excluded is inefficient in the sense of § 9.

* That \hat{P} may fluctuate is proved a fortiori by the non-existence of the maximum rate of profit (see § 7).

Nachwort

1) Viele Fragen sind noch zu klären; vieles was nur angedeutet werden konnte, bedarf noch der Ausführung. So ist etwa das Problem des Einflusses des Landes auf das Basissystem noch nicht befriedigend gelöst, während eine eingehende Untersuchung der erzwungenen Spezialisierung ("enforced specialisation") durch die Rente vielleicht noch interessante Resultate zeitigen könnte.

Dennoch ist so viel schon klar: Sraffa, der angeblich nur ein "Präludium zur Kritik der ökonomischen Theorie" schreiben wollte, hat selbst den Grundstein zu einer neuen Theorie gelegt, die zugleich durch den Reichtum der Begriffsbildungen und ihre Einheitlichkeit überrascht. Wenige Ökonomen hätten wohl gedacht, dass die Kuppelproduktion eine soviel differenziertere Struktur darstellt als die Einzelproduktion. Seit von Neumann war es wieder* üblich, das fixe Kapital als ein Kuppelprodukt aufzufassen. Aber erst aus Sraffas Buch konnte man sehen**, welche Möglichkeit einer fruchtbaren Ver-

* s.h. Sraffa Appendix D4 (/11/).

**Man vergleiche Dorfman-Samuelson-Solows Behandlung des von Neumannsystems: "One of the advantages of the von Neumannmodel is that it can handle capital goods without fuss and bother. A non depreciating capital good simply enters both as input and as output in the corresponding process. If the capital good depreciates 3 percent per unit of time, 1 unit of the good may appear as input and 0,97 unit as output." (/16, Seite 383). "Fixed capital" ist für die Autoren also nur der im Grunde untypische Spezialfall, den wir "Perennial Machines" genannt haben und der eigentlich keinen echten Fall der Kuppelproduktion darstellt.

einigung des makro- und des mikroökonomischen Standpunkts dieser Ansatz bietet.

Obwohl damit bei Sraffa einige Grundlagen zu einer realitätsbezogenen Analyse gegeben sind, ist noch eine immense theoretische Arbeit zu leisten, bevor man Production of Commodities by Means of Commodities in sinnvoller Weise zur Interpretation konkreter historischer Situationen heranziehen kann: unseres Erachtens gilt es, Sraffas Werk mit den Keynes'schen Vorstellungen über Geld, Zins und Beschäftigung in einem kohärenten System zu vereinigen und so die Grundlage für eine dynamische Akkumulation und Einkommensverteilung miteinbeziehende, General Theory zu schaffen. Wie in der Einführung bemerkt, weisen Joan Robinson und Kaldors Werk in dieser Richtung.

2) Walras' Modell des wirtschaftlichen Gleichgewichts und Sraffas Produktion der Waren durch Waren sind logisch gleichberechtigt, schliessen sich aber in der Anwendung ** im wesentlichen aus. Joan Robinson bemerkte schon in ihrer ersten Sraffa Rezension*: "The first is that, when we are provided with a set of technical equations for production and a real wage rate which is uniform throughout the

* Oxford Ec. Papers, February 1961 (zitiert nach /33/).

** Der Widerspruch zwischen der Theorie Sraffas und der der Produktionsfunktionen ist logischer Natur.

economy, there is no room for demand equations in the determination of equilibrium prices". ... "Some might complain that this is only flogging a dead Marshallian horse (which Sraffa himself helped to kill even before 1928*). But to my mind it emphasizes a point which, both in its scholastic and ist political aspect, is of great importance; in a market economy, either there may be a tendency towards uniformity of wages and the rate of profit in different lines of production, or prices may be governed by supply and demand, but not both**. Where supply and demand rule, there is no room for uniform levels of wages and the rate of profit. The Walrasian system makes sense, if we interpret it in terms of an artisan economy, where each producer is committed to a particular product, so that his income depends on his output and its price. Each can have a prospective rate of return on investment in his own line, but there is no mechanism to equalize profits between one line and another."

Bei Walras wird die Produktion durch die Zirkulation dominiert. Entsprechend können die Präferenzen der Wirtschaftssubjekte auch (Pareto-) optimal befriedigt werden. Dabei bleibt notwendigerweise offen, wie Ökonomie von Periode zu Periode sich entwickelt, wenn die Präferenzen ändern unter der Wirkung der in der vorigen Periode erzielten

*Dies spielt auf die erwähnten Artikel Sraffas von 1925 und 1926 an.

**Die Ausnahme zu dieser Regel wird diskutiert bei J. Schwartz (/35/ Part.III). Andere Kapitalgüter als was hier *Perennial Machines" heisst, (s.h. unten § 18), sind bei ihm bezeichnenderweise ausgeschlossen.

Produktionsresultate*.

In der "Produktion der Waren durch Waren" hingegen spielt die Zirkulation nur insofern eine Rolle, als vorausgesetzt ist, dass ein die Reproduktion ermöglichernder Tausch der Kapitalgüter stattfindet. Wie wir sehen werden, bereitet es Sraffa keine Schwierigkeiten, Kapitalgüter in sein System einzubeziehen, die während mehrerer Perioden mit wechselnder Produktivität im Gebrauch stehen. In der Theorie der Kuppelproduktion ergibt sich aus der uniformen Profitrate von selbst stets die richtige Abschreibungsformel für Maschinen, wie kompliziert auch die Veränderungen seien, denen sie während ihrer Lebensdauer unterworfen sind. Walras verwendet demgegenüber in seinem ursprünglichen Modell der Kapitalformation und des Kredits eine Formel für die Abschreibung, die nicht einmal bei Maschinen konstanter Effizienz und endlicher Lebensdauer korrekt ist**

Es mag sein, dass dieser Fehler heute behoben werden kann. Vorläufig sieht es aber so aus, als ob die Walras'sche Gleichgewichtsanalyse zur Beschreibung der Struktur einer sich reproduzierenden kapitalistischen Wirtschaft (und das heisst eben zuallererst zur Bestimmung des Werts

* Morishima (/26/), Seite 92.

** Dies hat Garegnani (/19/, Seite 93) bemerkt. Morishima, der als erster die Existenz einer Lösung für Walras' Modell der Kapitalformation und des Kredits bewiesen hat, ist diese Tatsache dagegen entgelaufen. Ob die über diesen speziellen Punkt hinausgehende Kritik Garegnanis an der Walras'schen Kapitaltheorie (op. cit. Seite 91-121) berechtigt ist, scheint noch offen. (s.h. die Uebersicht in M.Tiberi, /39/, Seite 7- 11).

von Kapitalgütern wechselnder Effizienz unter der Bedingung einer uniformen Profit- und Diskontrate) wenig beitragen könnte.

Hicks* kommt zum Schluss: "I have therefore come to the conclusion"... "that the Walrasian model had better be left, being useful enough in its own way, whenever our interest is in the horizontal structure of production, structure by industry,..." "But it should be matched" ..." by an approach from another angle, an Austrian approach, which does not pay attention to structure by industry, but fixes attention on time-sequence."¹ Demgegenüber möchte ich behaupten, dass in der vorliegenden Ausarbeitung von Sraffas Theorie des fixen Kapitals alle Resultate, die Hicks in seiner österreichischen Theorie gewinnt, abgeleitet worden sind, ohne dass deshalb der makroökonomische Gewichtspunkt oder die "horizontal structure of production" ignoriert werden müssen. Man mag darin einen Beweis sehen, dass Sraffas Modell dem Walras'schen als Theorie der Produktion und des Kapitals überlegen ist.² Grundsätzlich haben Walras und Sraffa dies gemein, dass sie eine klare Trennung** machen zwischen den technischen Produktionsbedingungen in Form von Input-Outputmatrizen mit gewissen Substitutionsmöglichkeiten (dies wäre bei Marx ein Ausdruck der "Produktivkräfte")

*Hicks (/22/). Es ist kennzeichnend, dass die "Fisher-Equation" bei uns eine Folge der Theorie ist (§ 16), während sie bei Hicks (op.cit., Seite 260) vorausgesetzt werden muss.

** vgl. Bhaduri. Bhaduri weist darauf hin, dass diese Trennung in der neoklassischen Theorie der Produktionsfunktionen verwischt wird, indem dort das technische Mass des Kapitals und sein Wertmass unkritisch identifiziert werden. Dass notwendigerweise eine Differenz zwischen beiden besteht, drückt sich bei Sraffa in der Abhängigkeit des Werts des Kapitalstocks von der Profitrate aus. (Bhaduri, /13/.)

und den sozio-ökonomischen Bedingungen, unter denen die Produktion steht (bei Marx ein Ausdruck der "Produktionsverhältnisse"). In der Darstellung der Produktivkräfte unterscheiden sie sich nur wenig, dagegen sehr in der der Produktionsverhältnisse: Bei Walras sind die Präferenzen einer umstrukturierten Menge von Wirtschaftssubjekten massgebend, die frei die vom Standpunkt der Transaktion gleichartigen Produktionsfaktoren tauschen. Anders bei Sraffa, wo vor den Präferenzen der Individuen eine uniforme Profitrate die Verteilung des Einkommens bestimmt, und wo die Preise der "Produktionsfaktoren" in ganz verschiedener Weise entstehen: der der Arbeit ist gegeben oder direkt aus der Profitrate abzuleiten, die Preise der Kapitalgüter sind im wesentlichen Kosten der Produktion; Rente ist ein Residuum.

Nun gibt es ohne Zweifel Aspekte des Wirtschaftslebens, wo der freie Tausch gleichberechtigter ökonomischer Agenten im Vordergrund steht. Sie sind aber nicht unbedingt dominierend. Für die Klassiker war der Reallohn auf der gleichen Ebene wie die übrigen Produktionsmittel gegeben; ins Mehrprodukt teilten sich die Kapitalisten mit den Grundbesitzern. Dabei spielten die Grundbesitzer eine passive Rolle, als unproduktive Konsumenten eines ihnen aus der Knappheit des zufällig verteilten Bodens entspringenden Residualeinkommens, während die Kapitalisten durch die Konkurrenz die Produktionsmethoden beständig aktiv revolutionierten und alle Möglichkeiten insofern gleichmäßig ausschöpften, als sie in der rastlosen Suche nach der profitabelsten Kapitalanlage den Profit in Proportion zum vorgesessenen Kapital verteilten. (Die Profitrate, dürfen wir hinzufügen, wurde zugleich zum Maßstab des Werts von früher investierten Kapitalen,

wie es die Bedingungen der Kuppelproduktion vorschreiben.) Die Keynesianische Theorie der Einkommensverteilung ist der klassischen ähnlich, insofern auch in ihr die dynamische Investitionstätigkeit der Kapitalisten (d.h. die Wachstumsrate g) bei gegebener Sparneigung die Profitrate determiniert - in der bekannten vereinfachten Formel, deren Begründung ^{**} wir hier nicht wiederholen (nur Kapitalisten sparen, d.h. $s = s_c$): $\pi = g/s$.

Die klassische und die neokeynesianische Vorstellungswelt sind im Grunde dynamischer Natur. Ihre Dynamik impliziert kausale Verknüpfungen, deren Wirkung auch am statischen Modell studiert werden kann. Beispiele sind der Multiplier oder eben Sraffas Produktion der Waren durch Waren, wo die Annahme einer die relativen Preise bestimmenden von aussen vorgegebenen Einkommensverteilung für eine Vereinfachung oder Abstraktion einer Fülle komplizierter dynamischer Zusammenhänge steht. Durch solche in der mathematischen Struktur nicht notwendig explizit ausgedrückte Relationen unterscheiden sich die Systeme* im tiefsten.

Die Dynamik, die hinter dem Walras'schen System gedacht wird, handelt primär von simultanem Austausch gleichartiger ökonomischer Agenten. Sraffa, der seine Theorie kritisch benutzt wissen will, vermeidet es,

*Wir können also C.C. von Weizsäcker nicht unbedingt beipflichten, wenn er schreibt "So können im wesentlichen alle Modelle Joan Robinsons und ihrer unmittelbaren Schüler als Spezialfälle der Arrow-Debreu-Annahmen aufgefasst werden" (/41/, Seite 97).

** s.h. z.B. Pasinetti: (/28/).

sich seinerseits auf eine bestimmte Hypothese über die Einkommensverteilung festzulegen*. Die klassische Hypothese des Klassenkampfs
ihm würde in grösstmöglichen Gegensatz zu Walras setzen, keine** könnte.
den Gegensatz überbrücken. Wie schon bemerkt, würden wir uns eine
Integration der Ideen Sraffas und der Neokeynesianischen Schule
wünschen.

3) Zusammenfassung

In Piero Sraffas "Produktion der Waren durch Waren" *** wird die Wirkung einer sich ändernden funktionalen Einkommensverteilung auf die Produktionspreise eines geschlossenen kapitalistischen Wirtschaftssystem betrachtet. In der vorliegenden Arbeit ist Sraffas Behandlung der Kuppelproduktion mathematisch streng dargestellt und erweitert worden. Es lässt sich feststellen, dass sich unter dem einheitlichen Begriff der Kuppelproduktion verschiednartige ökonomische

*Wenn man von der eher rätselhaften Bemerkung am Ende von § 41 (Sraffa /11/) und natürlich der negativen Bestimmung durch Ablehnung der Grenzproduktivitätstheorie absieht.

** Val. Schwartz, Part 3 (op.cit).

*** /11/

Kategorien verbergen. Deren Eigenschaften herauszuschälen, stellt die eigentliche Aufgabe dar.

Schon die einfache, aus der Theorie der Einzelproduktindustrien gebräuchige Unterscheidung von Basis und Nichtbasisgütern verlangt bei Kuppelproduktion eine differenziertere Untersuchung. Nur durch Berücksichtigung möglicher Komplikationen gelingt es, die Sraffa'sche Konstruktion des Basissystems durch einen Einzigkeitsbeweis im wesentlichen zu rechtfertigen. Der Beweis stützt sich auf eine mathematische Analyse der Bewegung der relativen Preise in Funktion der Profitrate (der Leser wird in der "erratischen" Bewegung dieser Preise die tiefere Ursache für die Möglichkeit einer "Wiederkehr der Techniken" und damit für die Unmöglichkeit der Ableitung einer neoklassischen Produktionsfunktion bei "regulären" Systemen erkennen*.)

Dann wendet sich die Untersuchung den Preisen selbst zu mit Fragen wie, inwieweit sie sich auf Arbeit zurückführen lassen, was negative Arbeitswerte bei Kuppelproduktion bedeuten, usw. Verschiedene Standards für Preise werden herangezogen, um die sich ändernde Verteilung zu beschreiben. Die Existenz des wichtigsten unter ihnen, Sraffas Standardware, wird dann genau geprüft. Die Suche nach der Standardware erweist sich als Äquivalent zur Suche nach der maximalen Profitrate.

* Die wird im Text nicht mehr ausgeführt; s.h. jedoch Fussnote im Anschluss an Satz 3.1, Seite 16.

eines Systems - und diese hängt wieder mit von Neumanns Expansionsfaktor für das System (betrachtet als von Neumann-System) zusammen. Damit kann die Existenz der Standardware und die Existenz positiver Preise zu bestimmten Effizienzkriterien in Bezug gesetzt werden.

Nach diesen sehr abstrakten Untersuchungen, die durch die Einführung einer neuen graphischen Technik nur geometrisch anschaulicher werden, ersetzen die konkreten Begriffe "fixes Kapital" und "Land" den allgemeineren der "Kuppelproduktion". In der Theorie des fixen Kapitals wird nachgewiesen, dass sich makroökonomisch durch die Einführung langlebiger Kapitalgüter wenig ändert im Vergleich zur Theorie der Einzelproduktindustrien, während mikroökonomisch Kapitalgüter sich in mannigfacher Weise differenzieren und klassifizieren lassen. Die Heranziehung des von Neumannmodells und bestimmter Effizienzkriterien zur Analyse der Wirkung von Änderungen der Profitrate auf die Preise liefert hier sehr konkrete Resultate.

Die Theorie der Rente und Quasirente schliesslich erweist sich als bedeutend schwieriger, jedenfalls bei den internen (von den externen zu unterscheidenden) fallenden Erträgen. Einerseits wird am Beispiel deutlich, dass die im ersten Teil bei der Theorie der Kuppelproduktion entwickelten Methoden nicht in jedem Fall zureichen, um die Existenz der Standardware in einem System das Land enthält nachzuweisen, andererseits zeigt die Theorie der "erzwungenen Spezialisierung", welche entscheidenden Einfluss die Rente auf den Gebrauch eines gegebenen, technisch zu verschiedenen Zwecken brauchbaren Landstücks haben muss. Es dürfte sich lohnen, der Theorie der Rente noch eine grössere Untersuchung zu widmen. Dies umso mehr, als die Theorie der

Rente als Theorie der Quasirenten auf geschützten Innovationen
auch in der Wachstumstheorie von Bedeutung ist.

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Note: Fairly complete bibliographies are to be found in

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CURRICULUM VITAE

Ich, Bertram Schefold, Bürger von Basel, bin in Basel am 28. 12. 1943 als Sohn des Dr. Karl Friedrich Schefold, Professor für Archäologie an der Universität Basel, und der Marianne Eleonore, geb. von den Steinen, geboren. Nach der Primarschule besuchte ich das Humanistische Gymnasium in Basel bis zur Matura (1962, Typus A). Dann studierte ich zuerst während drei Semestern in München (davon das erste als Hörer) und während zweien in Basel; am 29. 10. 1962 bestand ich das Vordiplom in Mathematik und Experimentalphysik in Basel. Im Wintersemester 1964/5 besuchte ich die Universität Hamburg, anschliessend wieder die Universität Basel bis zum Diplomexamen in Mathematik, Theoretische Physik und Philosophie am 3. 5. 1967 (Diplomarbeit über "Konforme Äquivalenz"). Im Sommersemester 1967 und im Wintersemester 1967/8 war ich in Basel immatrikuliert; als Präsident des Verbandes der Schweizerischen Studentenschaften blieb ich jedoch vom Besuch der Vorlesungen dispensiert. Im Sommer 1968 fing ich ein Zweitstudium in Nationalökonomie in Basel an und begann bei Professor Dr. G. Bombach die vorliegende Dissertation. Ich setzte das Oekonomiestudium in Cambridge, England, fort (1969 als Besucher der Faculty of Economics, 1970 als Advanced Student am King's College). Am 10. Juli 1971 bestand ich das Doktorexamen in den Fächern Nationalökonomie (Hauptfach), Philosophie und Mathematik (Nebenfächer). Im Wintersemester 1971/72 werde ich Lektor für mathematische Oekonomie an der Universität Basel.

Ich besuchte Vorlesungen und Uebungen in Basel bei den Herren Professoren E. Fink, K. Rossmann, H. Salmony, M. Thürkau (Philosophie); P. von der Mühll (Griechisch); G. Bombach, W. Kapp, J. Stohler (Nationalökonomie); P. Trappe, E. Salin (Soziologie); H. Guth (Statistik); M. Eichler, H. Huber, W. Habicht (Mathematik); P. Huber, E. Miescher, P. Scherrer, H. Striebel (Experimentalphysik); K. Alder (Theoretische Physik); A. Portmann (Zoologie); in Cambridge bei Frau Professor G. Anscombe (Philosophie); Frau Professor J. Robinson, und den Herren Prof. Lord Kahn, Prof. N. Kaldor, Dr. L. Pasinetti, R. Rowthorne, A. Singh (Oekonomie); in Hamburg bei PD Frau E. Ströker, Herrn Richter, und den Herren Professoren C.F. von Weizsäcker (Philosophie); L. Collatz, E. Kähler (Mathematik); in München bei den Herren Professoren R. Lauth, W. Stegmüller (Philosophie); H. Richter, K. Stein (Mathematik); A. Faessler (Experimentalphysik); H. Koppe (Theoretische Physik). Ihnen allen fühle ich mich zu grossem Dank verpflichtet.