Contract Theory - Intro

Roman Inderst

2017

Overview

- Focus is contract theory.
- We explore a principal - agent setting, with core applications to "firm - consumer", "firm - worker" and "lender - borrower".
- Conceptual problems:
  - Screening: Ex-ante informed agent.
  - Signaling: Ex-ante informed principal.
  - Moral Hazard: Interim private information.
- In addition, we explore extensions to behavioral contract theory.

Material

- Most relevant: Writing on board. Part is contained in lecture notes (notably, signaling and moral hazard).
- Additional reading: Next to 1-2 papers (will be announced), most relevant are chapters in any advanced micro economic textbook & in any advanced game theory book. In particular
  - Bolton and Dewatripont
  - Mas-Colell, Whinston, and Green
  - Fudenberg and Tirole
- Preparation for exam? Details will follow.
- **No questions regarding the exam and preparation for it will be answered outside classes!**

Contract Theory - Screening

Simple Pricing Problem

- Monopolist with constant cost c.
- Consumer types, for now two: z_l, z_h. Ex-ante: $q = \text{Pr}(z_h)$.
- Traded quantity is x. Total price paid / transfer t.
- Ex-ante private information / Adverse selection: Consumer privately knows z.

- Payoff functions: Consumer $U(x, t, z) = zu(x) - t$. Firm $V(x, t) = t - cx$.
- Benchmark: First- best solution ? [Board].
- General problem: Firm/Principal offers mechanism that the consumer/agent can then accept or reject.

Solving Restricted Problem

- We consider now the following "mechanism": Firm offers "incentive compatible menu of contracts".
  –> (x_l, t_l) and (x_h, t_h).

- What constraints must this satisfy? IF (!) both types shall accept:
  —> IC_l, IC_h, IR_l, IR_h [Board].

- Objective function for firm: [Board].

Solving the Problem

- Step 1: Why is IR_h redundant?
- Step 2: Why must IR_l bind?
- Step 3: Why must IC_h bind? [Otherwise increase t_h.]
- Step 4: $x_h \geq x_l$ [Proof: Combine the two ICs.]
- Step 5: Then, we can ignore IC_l.

Rewriting the Problem

- Remaining constraints: IR_l and IC_h.
- From this we have:

$$t_l = z_l u(x_l)$$

$$t_h = z_h u(x_h) - [z_h - z_l]u(x_l)$$

- Decomposition: Information rent.
- Now: 1) Rewrite the objective function. 2) Solve for optimal x_l, x_h.
- Interpretation? [Key words: Virtual surplus, no-distortion-at-the-top, information rent.]

Exercise at Home (1)

- Question 1: When is it indeed optimal for the firm to offer both types an acceptable contract?
- Question 2: Relate this to the concept of ”virtual surplus”.

Exercise at Home (2)

- Question 3: Suppose the reservation utility is not zero, but that each consumer can realize a given "outside option" value (from "consuming an outside good") of $u_0 > 0$. How do results change?

- Question 4: Suppose there are two other firms in the market, but they are at some "distance". Precisely, suppose:
  - Each consumer realizes utility $U_0(x, t, z) = zu(x) - t - y$ with $y > 0$ (but sufficiently small) when they buy quantity x at price t from one of the "distant firms"..
  - Suppose that the two distant firms compete a la Bertrand (zero profits) and thus offer "any unit at price equal to cost".

  Try to solve now the problem for the "former monopolist", given that now the firm's consumers have ("at a distance") what is called "type-dependent outside options / reservation values".

The Continuous Problem

- We now generalize the problem and we introduce a continuum of types.
- Agent utility again $U = u(x, z) + t$ (note: PLUS - this is the standard convention!)
- Principal utility now $V = v(x, z) - t$, which thus could depend on type (e.g. insurance problems).
- Let z be normalized so that in $[0, 1]$ with CDF $F(z)$ and density $f(z)$.
- Key assumptions (next to standard concavity assumptions and some technical assumptions):
  - $u(x, z)$ is strictly increasing in z.
  - Single Crossing Property:

$$\frac{\partial}{\partial z}\left[-\frac{\partial V/\partial x}{\partial V/\partial t}\right] = \frac{\partial^2 u(x, z)}{\partial x \partial z} > 0.$$

Procedure

- First-best benchmark?
- How to solve for "second best"? –> Program of principal?
  [Objective function, IC constraint(s), IR/PC constraint(s) – Which types?]
- "Old approach": Substitute constraints etc.
- "Modern / more intuitive approach": Directly appeal to implications of incentive compatibility / truth telling.
  – > We follow this!

Implications of IC

- Denote $U(z', z) = u(x(z'), z) + t(z')$. Now IC / TT holds when

$$z = \arg\max_{z'} U(z', z).$$

- Denote now the maximand (of the message) by $z' = z^*(z)$ (which by IC satisfies $z^*(z) = z$). So we can write the agent's utility under the mechanism/menu as

$$U = U(z^*(z), z).$$

  Then by the envelope theorem we have

$$\frac{dU}{dz} = \frac{\partial U}{\partial z} = \frac{\partial u}{\partial z}.$$

Rewriting the Program

- Note first that surely $U(z = 0) = 0$ (IR binds at the bottom), so that $U(z) = \int_0^z \frac{\partial u(x(\tau),\tau)}{\partial \tau} d\tau$, which is again the "information rent". [Board. Slope?]

- Using $U(z) = u(x(z), z) + t(z)$, we can write

$$t(z) = -u(x(z), z) + \int_0^z \frac{\partial u(x(\tau), \tau)}{\partial \tau} d\tau.$$

- Principal's problem rewritten (substitute out $t(z)$):

$$\max_{x(z)} \int_0^1 \left[v(x(z), z) + u(x(z), z) - \int_0^z \frac{\partial u(x(\tau), \tau)}{\partial \tau} d\tau \right] dF(z).$$

- Now transform $\int_0^1 \int_0^z \frac{\partial u(x(\tau),\tau)}{\partial \tau} d\tau dF(z)$ to (by partial integration)

$$\int_0^1 \frac{\partial u(x(z), z)}{\partial z} \frac{1 - F(z)}{f(z)} f(z) dz$$

Solution

- Ultimate program:

$$\max_{x(z)} \int_0^1 \left[v(x(z),z) + u(x(z),z) - \frac{\partial u(x(z),z)}{\partial z} \frac{1-F(z)}{f(z)} \right] f(z) dz.$$

- Pointwise maximization
  (using surplus function $s(x(z),z) = v(x(z),z) + u(x(z),z)$)

$$\frac{\partial s(x(z),z)}{\partial x} = \frac{1-F(z)}{f(z)} \frac{\partial^2 u(x(z),z)}{\partial z \partial x}.$$

- Implications ?

Extension 1: Mechanism Design

- Can the principal do better – with what game?
  –> To answer: Set up problem of ”mechanism design”

- For simplicity, return to pricing/monopoly problem.

- **Allocation:** $(x(z), t(z))$ = Function that assigns to each possible type of the agent a consumption quantity and a payment (in this context!).

- **Mechanism:** Is a game form, i.e., a set of strategies for the agent and a function $(x(s), t(s))$ that assigns to each possible strategy of the agent an outcome (x, t). [Thus, $s \in S$ are now strategies!]

- **A revelation mechanism:** Also called "direct mechanism". This is a mechanism $(x(z'), t(z'))$ in which the agent is asked to announce his type (message z') –> The agent's strategy space S is the set of types! And it has to be incentive compatible, so that $z' = z$ is optimal.

- **Implementation:** We say that a mechanism $[S, x(s), t(s)]$ implements an allocation $(x(z), t(z))$ if and only if for every type z there exists an optimal strategy $s^*(z)$ such that

$$(x(s^*(z)), t(s^*(z))) = (x(z), t(z)).$$

- **Revelation Principle:** An allocation $(x(z), t(z))$ can be implemented by some (arbitrarily complicated!) mechanism *if and only if* it can also be implemented by a (direct) revelation mechanism.

- **Intuition:** Consider an indirect mechanism where the buyer is privately informed about his type z and chooses from strategies $s \in S$. Suppose that this game results in an allocation $(x(s^*(z)), t(s^*(z)))$.

  By introducing a "mediator" we can construct a direct mechanism that results in the same allocation as the indirect mechanism:

  – The buyer reports the mediator some type z'.

  – The mediator plays $s^*(z')$ for the buyer who, consequently, receives payments as if he has played $s^*(z')$ himself. −> $x(s^*(z'))$ and $t(s^*(z'))$ are realized.

  In this new game it is optimal for the buyer to tell the mediator his true type z. If not, $s^*(z)$ would not have been optimal in the indirect mechanism.

- **General proof?** E.g. does the revelation principle hold with several agents? −> Check Fudenberg and Tirole or MWG. Read carefully, only notation!

Extension 2: The Insurance Problem

- Individuals are endowed with wealth W and can incur a loss $0 < L < W$.
- Individual's type $0 < p_l < p_h < 1$ determines probability of loss with $\Pr(p_i = p_l) = \alpha$.
- Individuals are risk-averse: $u'() > 0$ and $u''() < 0$
- The risk-neutral insurer offers contract (I, D) where I is the insurance premium and D is the deductible. If a loss occurs, the insurer pays the individual $L - D$. Thus, the individual who signs the contract obtains $W - I$ when no loss occurs and $W - D - I$, otherwise.

Questions:

- What will be the equilibrium contract in a monopoly?
- What will be the outcome in with (perfect) competition?

The Insurance Problem: Monopoly

- What happens when full information? -> full insurance and insurance premium equals the difference between initial wealth W and the certainty equivalent of the lottery without insurance.
- Asymmetric information: Individual privately knows his loss probability p_i.

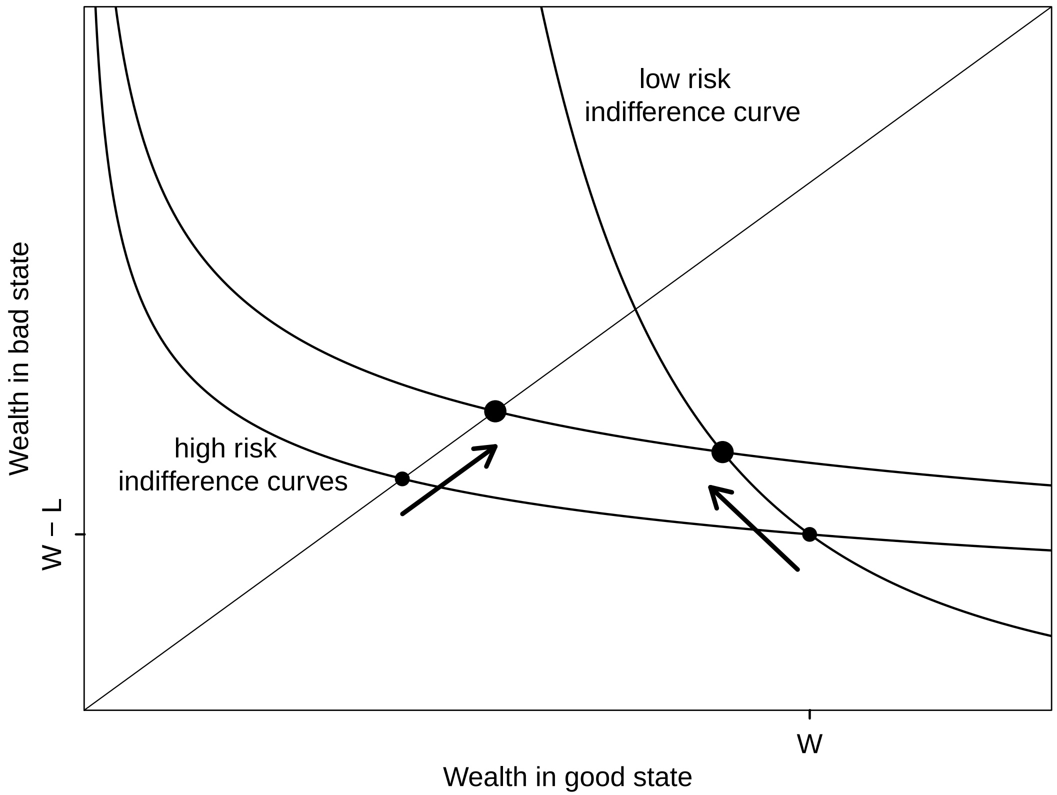
Solution: Standard monopolistic screening with (I_l, D_l) and (I_h, D_h) as choice variables.

- IR_h redundant: Role of the slope of indifference curve and risk aversion?
- IC_h and IR_l bind as I_h and I_l can be adjusted to make them bind.
- IC_l will be redundant. (check!)

Results:

- Full insurance at the top, partial insurance at the bottom.
- If α, the fraction of low risk types, is too low they may get no insurance.

The Insurance Problem: Monopoly



The Insurance Problem: Competition

Suppose now free entry -> zero profit condition.

What are candidates for equilibrium?

- Pooling equilibrium? Both types, p_l and p_h, obtain the same contract. –> New firm can enter and offer a new contract that attracts only low risk individuals and generates positive profits (why?). Thus, no pooling equilibrium can exist.

- Separating equilibrium? An eq'm candidate must fully insure the high risk type while making the low type's contract unattractive to the high type by providing partial insurance.

The Insurance Problem: Competition and Equilibrium Non-Existence

- In the above equilibrium the low risk types may still face a lot of risk.
- If there are only a few high risk types, a firm could enter offering a single full-insurance contract (pooling) that Pareto-dominates (why?) the separating contract.
- BUT: No pooling equilibrium exists!

–> If the fraction of high risk individuals, $(1 - \alpha)$, is sufficiently small there may be no equilibrium!

Solution?

Contract Theory - Signaling

Signaling and Adverse Selection: Repetition

- Simple game: The "Spence labour market signaling" game.
  - Worker has ability $\theta \geq 0$, which is privately known.
  - If firm employs worker, it realizes revenues of θ. Pays wage w.
  - Before "competition for the worker" ("break even"), the worker can try to signal his type by "costly education".
    –> Level y, comes at marginal cost $1/\theta$

$$\text{Utility: } u = w - y\frac{1}{\theta}.$$

- Single-crossing property in (w, y): $\frac{\partial u}{\partial w} = 1$, $\frac{\partial u}{\partial y} = -\frac{1}{\theta}$

$$\frac{dw}{dy} = -\frac{\partial u/\partial y}{\partial u/\partial w} = \frac{1}{\theta} \text{ –> } \frac{d}{d\theta}\left(-\frac{\partial u/\partial y}{\partial u/\partial w}\right) = -\frac{1}{\theta^2} < 0.$$

- **Rest on board !**

More Sophisticated Application: Signaling with Contracts

- Firm needs $I > 0$ (no own assets). Generates cash flows $x_l < x_h$.
  –> $\Pr(x = x_h) = p \in \{p_L, p_H\}$ with $p_L < p_H$.
  –> Denote $\Delta_x = x_h - x_l$ and $\Delta_p = p_H - p_L$.

- Prior: $q = \Pr(\theta = H)$. Denote $\widehat{p} := qp_H + (1-q)p_L$.

- Prior expected value is $V(\widehat{p}) = x_l + \widehat{p}\Delta_x$. For any p we have $V(p) = x_l + p\Delta_x$.

- The "financing game":
  - t=1: Firm offers $\mathbf{R} = \{R_l, R_h\}$ to investor(s).
  - t=2: Investor decides whether to accept or reject:
    –> Rejection: No financing.
    –> Acceptance: Firm receives I and invests.

Preliminary

- Game tree? Equilibrium concept: PBE.
- Bayes' rule "on equilibrium": Posterior beliefs

$$q(\mathbf{R}) = \mathsf{Pr}(\theta = H \mid \mathbf{R}) = \frac{\mathsf{Pr}(\theta = H \text{ and } \mathbf{R})}{\mathsf{Pr}(\mathbf{R})}.$$

–>Pooling and separating equilibrium candidates. (With respective $\mathbf{R}_L$ and $\mathbf{R}_H$.)

- Multiplicity problem arising from "free out-of-equilibrium beliefs": Illustration
  - Take "pessimistic" beliefs: For all $\mathbf{R} \notin \{\mathbf{R}_L, \mathbf{R}_H\}$ we have $q(\mathbf{R}) = \mathbf{0}$.
  - Easy to support pooling equilibria where investor makes **positive** profits! **How?**
- To restrict case distinctions: Suppose

$$V(p_L) < I \text{ but } V(\widehat{p}) > I.$$

–> This clearly implies $V(p_H) > I$.

- Key implication: Non-existence of separating equilibria.
  Why?

- Which pooling equilibria can we support? (Namely, with pessimistic beliefs?)
  - Note: If $q(\mathbf{R}) = \mathbf{0}$ then from $V(p_L) < I$ it follows that

$$R_h p_L + R_l(1 - p_L) < I\text{: No break-even!}$$

  - This allows to support **any** pooling equilibrium with

$$R_h \widehat{p} + R_l(1 - \widehat{p}) \geq I,$$

    where such values exist from $V(\widehat{p}) > I$.

- Take now the following particular security:

$$R_l = x_l \text{ and } R_h \text{ from } R_h \widehat{p} + x_l(1 - \widehat{p}) = I.$$

- This minimizes *information sensitivity*.
  $->$ Maximum payoff for H-type!

- Definition of "information sensitivity": Denote security value by

$$S(p) := R_h p + R_l(1 - p).$$

  $->$ We can express break-even requirement as

$$qS(p_H) + (1 - q)S(p_L) = I.$$

$$S(p_L) + q\left[S(p_H) - S(p_L)\right] = I.$$

  – Least "information sensitive" security *minimizes* pricing differential

$$S(p_H) - S(p_L) = (p_H - p_L)\Delta_R,$$

  namely by minimizing the security's upside Δ_R.

- Why does this maximize the payoff of the H-type? *Underpricing*

$$\begin{aligned} S(p_H) - S(\widehat{p}) &= (p_H - \widehat{p})\Delta_R \\ &= (1-q)(p_H - p_L)\Delta_R. \end{aligned}$$

- Back to "the game":
  –> First restrict consideration to the set of candidate equilibria where investors just break even, i.e., where

$$\begin{aligned} 0 &\leq R_l \leq x_l, \\ R_h &= \frac{I - R_l(1-\widehat{p})}{\widehat{p}}. \end{aligned}$$

  –> Note again: Contract with $R_l = x_l$ minimizes R_h!

- **Now to show: Among the BE-contract set, only contract with $R_l = x_l$ survives the Intuitive Criterion.**

Refinement with Security Design

- Take any other such contract with $R_l < x_l$.
  –> I.e., candidate equilibrium with $\mathbf{R}_L = \mathbf{R}_H = \mathbf{R} = (R_l < x_l, R_h)$, where $R_h = \frac{I - R_l(1-\widehat{p})}{\widehat{p}}$ ("break even").

- Consider now "out-of-equilibrium" offer with $\widehat{R}_l = R_l + d_l$ and $\widehat{R}_h = R_h - d_h$ such that L-type strictly prefers candidate equilibrium contract:

$$p_L(x_h - R_h) + (1 - p_L)(x_l - R_l) > p_L(x_h - R_h + d_h) + (1 - p_L)(x_l - R_l - d_l)$$

$$(1 - p_L)d_l > p_L d_h.$$

–> Consequence: L-type strictly worse off with $(x_{l,} \widehat{R}_h)$ *regardless* of investor's out-of-equilibrium beliefs and his (best) response ("accept/reject")!

- Consequence for refinement?
  –> $q(\widehat{\mathbf{R}}) = 1$! (No mass on L type!)

- Can we now adjust (d_l, d_h) such that $(1 - p_L)d_l > p_L d_h$ *and:*
  - Investor break even with $q(\widehat{\mathbf{R}}) = 1$;
  - H-type strictly prefers $\widehat{\mathbf{R}}$, given that accepted?

- Take (d_l, d_h) such that investor would be indifferent under "average beliefs"

$$\widehat{p}R_h + (1 - \widehat{p})R_l = \widehat{p}(R_h - d_h) + (1 - \widehat{p})(R_l + d_l)$$

$$(1 - \widehat{p})d_l = \widehat{p}d_h,$$

  - which from $\widehat{p} > p_L$ implies $(1 - p_L)d_l > p_L d_h$,
  - and makes H better off: $(1 - p_H)d_l > p_H d_h$.

- Final comments:
  - Intuitive Criterion has no "bite" to rule out all contracts with

$$R_l = x_l \text{ and } R_h > \frac{I - x_l(1 - \widehat{p})}{\widehat{p}}.$$

    Why?

  - We need a stronger requirement: (Universal) Divinity
    –> See Nachman and Noe (1994).

Contract Theory - Moral Hazard I

Introduction

- Most simple setting:
  - Principal vs. agent.
  - Agent's action e (effort) is not contractible, results in effort costs $\psi(e)(= e^2/k)$.
  - For principal: High output $x_h > x_l$ with probability $p(e) = e$.
- Game of TIOLI offer. Questions:
  - First best?
  - How can be implemented (”selling firm”)?
- Now add realism/complexity: Limited liability.

The Two-Outcome Case

- Contracting problem: Wages $w_h, w_l \geq 0$.
- Principal's maximization problem? The "second-best" solution. **[Board]**

Continuous Action and Outcomes

- Continuous outcome distribution $x \geq 0$.
- Affected by agent effort: CDF $H(x|e)$. Shift in the sense of Monotone Likelihood Ratio Property (MLRP).
- Disutility of higher effort: $\psi(e)$ with $\psi' > 0$.
- Contract: Payment to agent $w(x)$.

MLRP as a Modelling Tool

- Take $x'' > x'$ and $e'' > e'$, then

$$\frac{h(x''|e'')}{h(x'|e'')} > \frac{h(x''|e')}{h(x'|e')}.$$

- Equivalent

$$\frac{d}{dx}\left[\frac{\frac{\partial h(x|e)}{\partial e}}{h(x|e)}\right] > 0.$$

Digression: Information Orderings

- FOR THE MOMENT ONLY: Consider the following LEARNING PROBLEM
  - State e generated by some CDF $F(e)$ with density $f(e)$.
  - We learn about the state e by observing a signal x, with "signal-generating distribution" $h(x|e)$.

- (Bayesian) updating

$$\text{Posterior: } g(e \mid x) = \frac{h(x|e)f(e)}{\int h(x|e')f(e')de'}.$$

–> MLRP ensures that $G(e \mid x)$ satisfies "First-Order Stochastic Dominance" (FOSD) in x.

FOSD

- We have now a "family of distributions $G(e \mid x)$" (indexed by x!).
- FOSD: With $x'' > x'$ we have $G(e \mid x'') < G(e \mid x')$ at interior of the support of e.
  –> This just means that $G(e \mid x'')$ places more probability mass at higher realizations of e!
- A key implication is that for any non-decreasing (and somewhere strictly increasing) function $\pi(e)$ we have that

$$\int \pi(e)g(e \mid x'')de > \int \pi(e)g(e \mid x')de.$$

–> This is why "higher x are good news on e" (if we care for higher e!)

Second-order Stochastic Dominance

- SOSD orders distributions according to "riskiness/dispersion".
- $G(e|x')$ second-order stochastic dominates $G(e|x'')$ if and only if for all non-decreasing and concave $\pi(e)$ we have

$$\int \pi(e)g(e \mid x')de > \int \pi(e)g(e \mid x'')de.$$

–> Risk-averse agents "dislike" riskier distributions.

- This is equivalent to

$$\int_{-\infty}^{s} G(e|x'')de \geq \int_{-\infty}^{s} G(e|x')de$$

for all s with strict inequality for some s.

- Relation to mean-preserving spread and an elementary increase in risk?

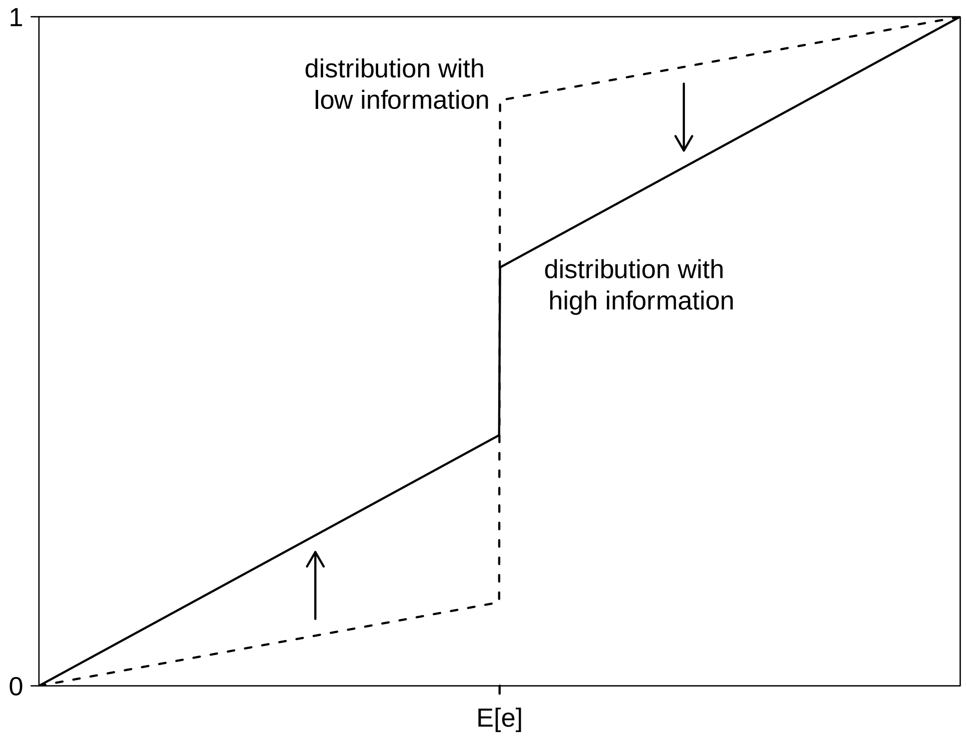
Application: Value of Information

- Risk-neutral agent has to choose between options A and B.
- A has unknown value $\widetilde{v}$ distributed on $[0, \overline{v}]$ with cdf $G(v)$, B has a safe payoff $\widehat{v}$.
- An information level of t is obtained at costs $c(t)$ with $c' > 0$ and $c'' > 0$. Denote with $G(v|t)$ the ex-ante posterior distribution of A's value given information t.

How to model learning?

- More information (higher t) leads to higher dispersion of $G(v|t)$. Why?
- In order to obtain nice comparative statics model learning as "rotation"!
- **Rotation**: $G(v|t)$ increases in t for $v < E[v]$ and decreases in t for $v > E[v]$.

Illustration: Suppose $G(v)$ is uniform and t is the probability that the agent learns the true value of A.



How much will the consumer learn? How is this affected by the value of his safe option $\widehat{v}$?

Expected value of agent (through integration by parts):

$$V = G(\widehat{v}|t)\widehat{v} + \int_{\widehat{v}}^{\overline{v}} v g(v|t) dv - c(t) = \overline{v} - \int_{\widehat{v}}^{\overline{v}} G(v|t) dv - c(t).$$

First order condition wrt t:

$$-\int_{\widehat{v}}^{\overline{v}} \left. \frac{\partial G(v|t)}{\partial t} \right|_{t=t^*} dv = c'(t^*).$$

- How does the optimal amount of information t^* change in $\widehat{v}$?
- The sign of $\partial t^* / \partial \widehat{v}$ depends on $\partial^2 V / \partial t \partial \widehat{v}|_{t=t^*}$ (implicit differentiation!):

$$\frac{\partial^2 V}{\partial t \partial \widehat{v}} = \left. \frac{\partial G(v|t)}{\partial t} \right|_{t=t^*, v=\widehat{v}}$$

In turn, the sign of this expression follows immediately from the rotation property!

$->$ Where is the value of information highest?

Back to MH Model with Continuum

- The moral-hazard problem: "The first-order approach"
  –> For given $w(x)$ agent's effort is pinned down by the first-order condition

$$\int_X w(x)\frac{\partial h(x|e^*)}{\partial e^*}dx - \psi'(e^*) = 0.$$

- Different way to set up problem: Who makes offer / has contractual power?
- Suppose here **agent** makes offer (cf. later in bargaining setting - general framework): Subject to the constraint

$$\int_X [x - w(x)]h(x|e^*)dx \geq I.$$

- Agent's objective function is to choose $w(x)$ so as to maximize

$$\int_X w(x)h(x|e^*)dx - \psi(e^*).$$

- The way to solve this is via a Lagrange approach!

- Lagrangian is

$$\begin{aligned} L &= \int_X w(x)h(x|e^*)dx - \psi(e^*) \\ &\quad +\mu \left[\int_X w(x) \frac{\partial h(x|e^*)}{\partial e^*} dx - \psi'(e^*) \right] \\ &\quad +\lambda \left[\int_X \left[x - w(x) \right] h(x|e^*)dx - I \right]. \end{aligned}$$

- This is maximized "pointwise" (for every x).

- Reformulating: $L =$

$$\int_X w(x) \left[1 + \mu \frac{\frac{\partial h(x|e^*)}{\partial e^*}}{h(x|e^*)} - \lambda \right] h(x|e^*) dx$$
$$+\lambda \int_X x h(x|e^*) dx - \psi(e^*) - \mu \psi'(e^*) - \lambda I.$$

- Note: We are only interested in the terms multiplied by $w(x)$.
  –> Pointwise maximization? ("Bang-bang")
  –> Order x according to values of multiplier, then $w(x) = 0$ or $w(x) =$?? [Maximum ?]
  –> Impose restriction $w(x) \leq x$!

- Precisely:

$$w(x) = \begin{cases} 0 & \text{if } 1 + \mu \frac{\frac{\partial h(x|e^*)}{\partial e^*}}{h(x|e^*)} - \lambda < 0 \\ x & \text{if } 1 + \mu \frac{\frac{\partial h(x|e^*)}{\partial e^*}}{h(x|e^*)} - \lambda > 0 \end{cases} .$$

- Note now that MLRP is equivalent to

$$\frac{d}{dx} \left[\frac{\frac{\partial h(x|e^*)}{\partial e^*}}{h(x|e^*)} \right] > 0.$$

–> *Result:* $w(x) = 0$ up to some cutoff x', then $w(x) = x$!

- Making the contract more realistic?

  –> Continous?

  –> Slope restriction: $x - w(x)$ nondecrasing?

- What is then the optimal contract?

Contract Theory - Moral Hazard II

Channel Management Application (1)

- Problem 1: Firm with cost c sets price p. Downstream demand $q(p)(= 1 - p)$.
- Problem 2: Channel management
  - Stage 1: Firm sets wholesale price w at which retailer can buy
  - Stage 2: Given w, retailer sets price p (facing demand $q(p)$).
- [Solution on board] [Discuss source of inefficiency]

Channel Management Application (2)

- Solutions in this setting? Non-linear contracts
  - "Two-part tariff": (F, w) -> Solution?
  - "Quantity forcing": (F, q^*) -> Solution?
- Adding realism (and bringing this back into our problems):
  - Screening: Retailer privately observes demand shock ("intercept").
  - Moral hazard: Price p not observable/contractible, but only final quantity. Addition of demand shock.

Two-State Problem - But as Two-sided Moral Hazard

- Application: Two-sided moral hazard problem
  –> Used in the large literature on "venture capital contracting".
- Core model from Inderst/Müller JFE 2004
  - Penniless entrepreneur needs $I > 0$.
  - Payoff x_l with probability $1 - p$, and x_h with probability p.
    –> Where $x_l < I < x_h$. Set $x_l = 0$!
  - Two-sided moral hazard problem: Effort e of entrepreneur, effort a of venture capitalist (VC) –> $p(e, a)$ with $p_e > 0$ and $p_a > 0$.
- Common functional specifications:
  - Linear: $p(e, a) = \gamma a + (1 - \gamma)e$.
  - Cobb-Douglas: $p(e, a) = a^{\gamma} e^{1-\gamma}$.

Two-sided Moral Hazard Problem

- Note: With only two outcomes, $x_l = 0 < x_h$, incentives for hard work depend only on share of the high outcome (generally, the "upside")
  –> VC's share is s, entrepreneur's share is $1 - s$.

- Effort costs are $c(e)$ and $g(a)$.
  –> For explicit solutions: Quadratic with

$$c(e) = \frac{e^2}{2\alpha_E} \text{ and } g(a) = \frac{a^2}{2\alpha_{VC}}.$$

- First-order conditions for respective effort:

$$\begin{aligned} s p_a(e^*, a^*) x_h - g'(a^*) &= 0, \\ (1 - s) p_e(e^*, a^*) x_h - c'(e^*) &= 0. \end{aligned}$$

- Take now the linear technology: $p(e,a) = \gamma a + (1-\gamma)e$.
  - Effort levels:

$$\begin{aligned} a^*(s) &= s\left[\alpha_{VC}\gamma x_h\right], \\ e^*(s) &= (1-s)\left[\alpha_E(1-\gamma)x_h\right]. \end{aligned}$$

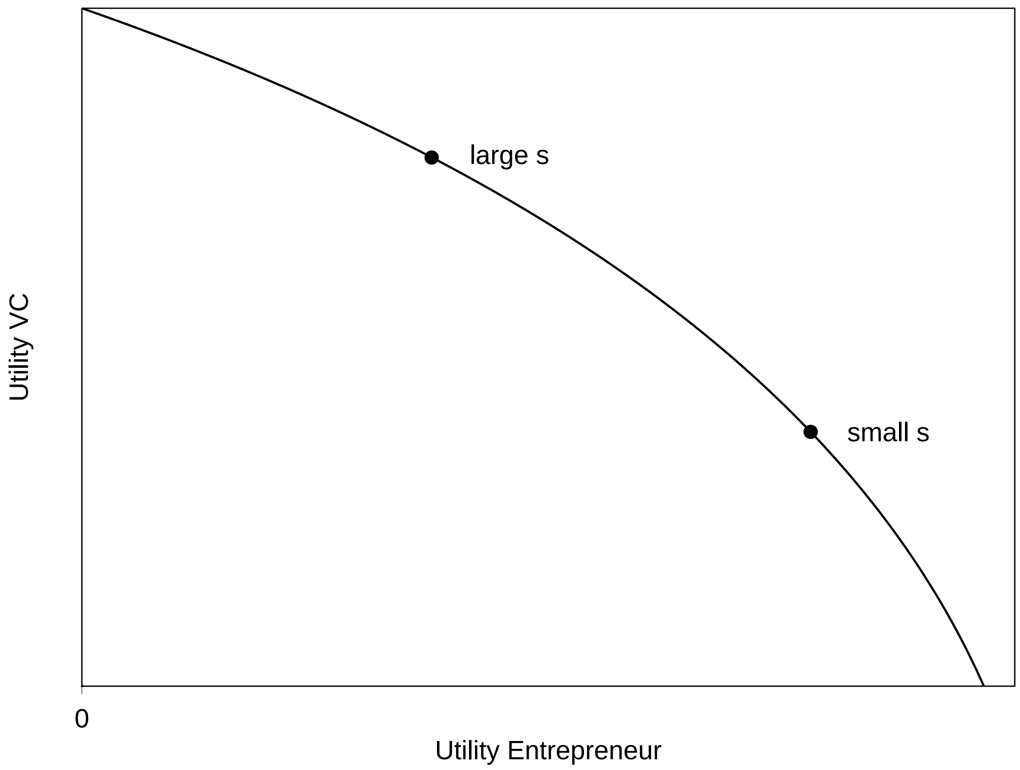
  - Generates

$$\begin{aligned} p^*(s) &= \gamma a^* + (1-\gamma)e^* \\ &= x_h\left[s\gamma^2\alpha_{VC} + (1-s)\alpha_E(1-\gamma)^2\right]. \end{aligned}$$

- Assume that VC has *no other funds*.
  –> Thus financial contract is *fully described* by sharing rule $(s, 1-s)$!
  –> (Gross) utilities are thus fully described by

$$\begin{aligned} u_E(s) &= (1-s)p(e^*,a^*)x_h - c(e^*), \\ u_{VC}(s) &= sp(e^*,a^*)x_h - g(a^*). \end{aligned}$$

Graphical Exposition



General Exposition of Contracting Power

- Who makes offer? What are the other party's outside options?
- How does this affect the contract (and efficiency)?

Introduction to Bargaining Theory

- **Nash Bargaining Solution** (for two-player bargaining games):
  - Axiomatic approach ("Co-operative game theory")
  - Strategic approach ("Non-cooperative game theory")
- Definition of an axiomatic bargaining problem $F : (U, D) \rightarrow S$.
  - U is a subset of R^2 and denotes the set of all feasible "utility pairs" (U_1, U_2) for the two players.
    –> E.g.: Split sum of one. Then $U_1 = u_1(x)$ and $U_2 = u_2(1 - x)$.
  - D is the disagreement point. E.g., $u_1(0)$ and $u_2(0)$.
  - S is the set of all solutions.

Axiomatic Nash Bargaining Solution

- Four Axioms:
  1. Pareto optimality ("on bargaining frontier")
  2. Symmetry (if symmetric problem, then symmetric solution)
  3. Invariance to affine transformations ($T(x) = Ax + b$)
     Eg. Scaling up outside option and bargaining set
     –> Then: Solution scaled up by the same factor
  4. Independence of Irrelevant Alternatives
- Then unique solution: Define "frontier function" $U_2 = \psi(U_1)$.
  –> Unique solution maximizes the (generalized) "Nash product"

$$[U_1 - D_1]^{\eta} [\psi(U_1) - D_2]^{1-\eta} .$$

Axiomatic NBS (Application)

- Solving for the first-order condition:

$$\beta := \frac{\eta}{1-\eta} = \left[-\psi'(U_1^*)\right] \frac{U_1^* - D_1}{\psi(U_1^*) - D_2}.$$

- **Trivial case:** Two risk-neutral players share "one Euro"
  - $U_2 = 1 - U_1$, i.e., $\psi' = -1$. Thus

$$\beta := \frac{\eta}{1-\eta} = \frac{U_1 - D_1}{(1 - U_1) - D_2}.$$

  - We can also solve this as: Player 1 receives D_1 plus fraction η of the "net surplus", which is $1 - D_1 - D_2$ —>

$$U_1^* = D_1 + \eta \left[1 - D_1 - D_2\right].$$

[Back to the MH Problem]

- **What is the bargaining set in our problem?**
  All pairs (u_E, u_{VC}) that are generated by varying s!

$$u_{VC}(s) = (x_h)^2\, s \left[\frac{1}{2} s\gamma^2 \alpha_{VC} + (1-s)\alpha_E(1-\gamma)^2\right],$$

$$u_E(s) = (x_h)^2\,(1-s) \left[\frac{1}{2}(1-s)\alpha_E(1-\gamma)^2 + s\gamma^2\alpha_{VC}\right].$$

  Note: $u_{VC}(s)$ is not necessarily maximized at $s = 1$!

- Precisely: If

$$\alpha_E(1-\gamma)^2 > \gamma^2\alpha_{VC},$$

then only $s \le \overline{s}$ are *Pareto optimal*, where

$$\overline{s} = \frac{\alpha_E(1-\gamma)^2}{2\alpha_E(1-\gamma)^2 - \gamma^2\alpha_{VC}}.$$

- What is the shape of the "Pareto/efficiency" frontier?
- How are the equity stakes determined by "market forces"?
  –> Modeling tool: Outside/breakdown options!
- **Further illustration on the board, including embedding in (search) market framework!**
  [Contracts with bargaining in a market framework "at work"...]