thereof) must be self-sustaining rather than contracted upon. The basic mechanism is otherwise similar to the Bolton–Scharfstein mechanism, in that lenders cannot appropriate any of the current return and count solely on the nonrefinancing threat to recoup their investment. Bulow and Rogoff consider an infinite-horizon, symmetric-information model in which (a) the sovereign can decide not to reimburse and (b) the sovereign can save, and (c) the rate of growth of the economy is smaller than the rate of interest. They show that no lending is feasible as the borrower always prefers to default (and save some of the concomitant extra income).

Several contributions have shown that borrowing may be feasible in more general no-sanction environments. First, Hellwig and Lorenzoni (2004) show that when the rate of growth in the absence of sovereign borrowing exceeds the rate of interest, then sovereign debt borrowing is feasible, even though incentive-compatible repayments still require borrowing levels below the first-best level. Intuitively, exclusion from borrowing is a stronger threat when the rate of growth is large relative to the rate of interest. Second, an outright exclusion, in which the defaulting sovereign cannot even save, makes it particularly costly for the sovereign to repudiate its debt. Again, some sovereign debt may then be issued in equilibrium (Kehoe and Levine 1993; Kocherlakota 1996). Finally, standard "type-based" reputation models (see, for example, Kreps et al. 1982) would deliver some equilibrium borrowing.

3.9 Exercises

Exercise 3.1 (random financing). Consider the fixed-investment model of Section 3.2. We know that if $A \geqslant \bar{A}$, where

$$I - \bar{A} = p_{\mathrm{H}}\left(R - \frac{B}{\Delta p}\right),$$

it is both optimal and feasible for the borrower to sign a contract in which the project is undertaken for certain. We also noted that for $A < \bar{A}$, the borrower cannot convince investors to undertake the project with probability 1. With $A > 0$, the entrepreneur benefits from signing a "random financing contract," though.

(i) Consider a contract in which the borrower invests $\hat{A} \in [0, A]$ of her own money, the project is financed with probability x, and the borrower receives R_{b} in the case of success and 0 otherwise. Write the investors' breakeven condition.

(ii) Show that (provided the NPV, $p_{\mathrm{H}}R - I$, is positive) it is optimal for the borrower to invest

$$\hat{A} = A.$$

How does the probability that the project is undertaken vary with A?

Exercise 3.2 (impact of entrepreneurial risk aversion). Consider the fixed-investment model developed in this chapter: an entrepreneur has cash amount A and wants to invest $I > A$ into a project. The project yields $R > 0$ with probability p and 0 with probability $1 - p$. The probability of success is p_{H} if the entrepreneur works and $p_{\mathrm{L}} = p_{\mathrm{H}} - \Delta p$ ($\Delta p > 0$) if she shirks. The entrepreneur obtains private benefit B if she shirks and 0 otherwise. Assume that

$$I > p_{\mathrm{H}}\left(R - \frac{B}{\Delta p}\right).$$

(Suppose that $p_{\mathrm{L}}R + B < I$; so the project is not financed if the entrepreneur shirks.)

(i) In contrast with the risk-neutrality assumption of this chapter, assume that the entrepreneur has utility for consumption c:

$$u(c) = \begin{cases} c & \text{if } c \geqslant c_0, \\ -\infty & \text{otherwise.} \end{cases}$$

(Assume that $A \geqslant c_0$ to ensure that the entrepreneur is not in the "$-\infty$ range" in the absence of financing.)

Compute the minimum equity level $\bar{A}$ for which the project is financed by risk-neutral investors when the market rate of interest is 0. Discuss the difference between $p_{\mathrm{H}} = 1$ and $p_{\mathrm{H}} < 1$.

(ii) Generalize the analysis to risk aversion. Let $u(c)$ denote the entrepreneur's utility from consumption with $u' > 0$, $u'' < 0$. Conduct the analysis assuming either limited liability or the absence of limited liability.

Exercise 3.3 (random private benefits). Consider the variable-investment model: an entrepreneur initially has cash A. For investment I, the project yields RI in the case of success and 0 in the case of failure.

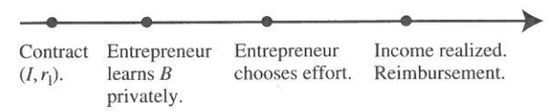

Figure 3.10

The probability of success is equal to $p_{\mathrm{H}} \in (0,1)$ if the entrepreneur works and $p_{\mathrm{L}} = 0$ if the entrepreneur shirks. The entrepreneur obtains private benefit BI when shirking and 0 when working. The per-unit private benefit B is unknown to all *ex ante* and is drawn from (common knowledge) uniform distribution F:

$$\Pr(B < \hat{B}) = F(\hat{B}) = \hat{B}/R \quad \text{for } \hat{B} \leqslant R,$$

with density $f(\hat{B}) = 1/R$. The entrepreneur borrows $I - A$ and pays back $R_{\mathrm{l}} = r_{\mathrm{l}}I$ in the case of success. The timing is described in Figure 3.10.

(i) For a given contract (I, r_{l}), what is the threshold B^*, i.e., the value of the private per-unit benefit above which the entrepreneur shirks?

(ii) For a given B^* (or equivalently r_{l}, which determines B^*), what is the debt capacity? For which value of B^* (or r_{l}) is this debt capacity highest?

(iii) Determine the entrepreneur's expected utility for a given B^*. Show that the contract that is optimal for the entrepreneur (subject to the investors breaking even) satisfies

$$\tfrac{1}{2}p_{\mathrm{H}}R < B^* < p_{\mathrm{H}}R.$$

Interpret this result.

(iv) Suppose now that the private benefit B is observable and verifiable. Determine the optimal contract between the entrepreneur and the investors (note that the reimbursement can now be made contingent on the level of private benefits: $R_{\mathrm{l}} = r_{\mathrm{l}}(B)I$).

Exercise 3.4 (product-market competition and financing). Two firms, $i = 1, 2$, compete for a new market. To enter the market, a firm must develop a new technology. It must invest (a fixed amount) I. Each firm is run by an entrepreneur. Entrepreneur i has initial cash $A_i < I$. The entrepreneurs must borrow from investors at expected rate of interest 0. As in the single-firm model, an entrepreneur enjoys private benefit B from shirking and (
The probability of success is p_{H} an
when working and shirking.

The return for a firm is

$$R = \begin{cases} D & \text{if both firms succeed in d} \\ & \text{technology (which results} \\ M & \text{if only this firm succeeds} \\ & \text{enjoys a monopoly situati} \\ 0 & \text{if the firm fails,} \end{cases}$$

where $M > D > 0$.

Assume that $p_{\mathrm{H}}(M - B/\Delta p) < I$
Nash equilibrium in contracts (wh
neur negotiates with investors, b
rectly anticipate whether the othe
obtains funding). In a first step, assu
firms' projects or research technolog
dent, so that nothing is learned fron
failure of the other firm concerning
the borrower.

(i) Show that there is a cutoff $\underline{A}$ suc
entrepreneur i obtains no funding.

(ii) Show that there is a cutoff $\bar{A}$ su
for $i = 1, 2$, both firms receive fundi

(iii) Show that if $\underline{A} < A_i < \bar{A}$ for $i =$
exist two (pure-strategy) equilibria.

(iv) The previous questions ha
when investment projects are indepe
market competition makes it *more*
entrepreneur to obtain financing. L
that when projects are correlated,
competition may *facilitate financi*
financiers to benchmark the entrep
mance on that of competing firms.

Let us change the entrepreneu
slightly:

$$u(c) = \begin{cases} c & \text{if } c \geqslant c_0, \\ -\infty & \text{otherwis} \end{cases}$$

That is, the entrepreneur is infinitel
low c_0 (this assumption is stronger t
it simplifies the computations).

Suppose, first, that only one firm
that the necessary and sufficient c
vestment to take place is

$$p_{\mathrm{H}}\left(M - \frac{B}{\Delta p}\right) - c_0 \geqslant I -$$

g on from question (iv), suppose now wo firms and that their technologies *orrelated* in that if both invest and eurs work, then they both succeed or he technically oriented reader, there lying state variable ω distributed uni- | and common to both firms such that succeeds if $\omega < p_L$, always fails if ucceeds if and only if the entrepre- en $p_L < \omega < p_H$.)

$$p_H D - c_0 \geqslant I - A,$$

uilibrium for both entrepreneurs to . Conclude that product-market com- cilitate financing.

ontinuous investment and decreas- scale). Consider the continuous- odel, with one modification: invest- return $R(I)$ in the case of success, se of failure, where $R' > 0$, $R'' < 0$, $R'(\infty) < 1/p_H$. The rest of the model The entrepreneur starts with cash A. of success is p_H if the entrepreneur $= p_H - \Delta p$ if she misbehaves. The btains private benefit BI if she mis- otherwise. Only the final outcome is t I^* denote the level of investment total surplus: $p_H R'(I^*) = 1$. investment $I(A)$ vary with assets? the shadow value v of assets (the de- borrower's gross utility with respect with the level of assets?

enegotiation and debt forgiveness). ng the multiplier k (given by equation assumed that it is optimal to spec- the borrower large enough that the raint (IC_b) is satisfied. Because condi- es that the project has negative NPV misbehavior, such a specification is when the contract cannot be renego- pose of this exercise is to check in a cal way that the borrower cannot gain oan agreement in which (IC_b) is not which is potentially renegotiated be- ver chooses her effort. While there is ay to prove this result, some insights are gleaned from this pedestrian approach. Indeed, the exercise provides conditions under which the lender is willing to forgive debt in order to boost incentives (the analysis will bear some resemblance to that of liquidity shocks in Chapter 5, except that the lender's concession takes the form of debt forgiveness rather than cash infusion).[60]

(i) Consider a loan agreement specifying investment I and stake $R_b < BI/\Delta p$ for the borrower. Suppose that the loan agreement can be renegotiated after it is signed and the investment is sunk and before the borrower chooses her effort. Renegotiation takes place if and only if it is mutually advantageous. Show that the loan agreement is renegotiated if and only if

$$(\Delta p)RI - \frac{p_H BI}{\Delta p} + p_L R_b \geqslant 0.$$

(ii) Interpret the previous condition. In particular, show that it can be obtained directly from the general theory. Hint: consider a fictitious, "fixed-investment" project with income $(\Delta p)RI$, investment 0, and cash on hand $p_L R_b$.

(iii) Assume for instance that the entrepreneur makes a take-it-or-leave-it offer in the renegotiation (that is, the entrepreneur has the bargaining power). Compute the borrowing capacity when $R_b < BI/\Delta p$ and the loan agreement is renegotiated.

(iv) Use a direct, rational expectations argument to point out in a different way that there is no loss of generality in assuming $R_b \geqslant BI/\Delta p$ (and therefore no renegotiation).

Exercise 3.7 (strategic leverage). (i) A borrower has assets A and must find financing for an investment $I(\tau) > A$. As usual, the project yields R (success) or 0 (failure). The borrower is protected by limited liability. The probability of success is $p_H + \tau$ or $p_L + \tau$, depending on whether the borrower works or shirks, with $\Delta p = p_H - p_L > 0$. There is no private benefit when working and private benefit B when shirking. The financial market is competitive and the expected rate of return demanded by investors is equal to 0. It is never optimal to give incentives to shirk.

60. The phenomenon of debt renegotiation has been analyzed in a number of settings: see, for example, Bulow and Rogoff (1989a,b), Eaton and Gersovitz (1981), Fernandez and Rosenthal (1990), Gale and Hellwig (1989), Gromb (1994), Hart and Moore (1989, 1995), and Snyder (1994).

The investment cost I is an increasing and convex function of τ (it will be further assumed that $p_H R > I(0)$, that in the relevant range $p_H + \tau < 1$, and that $I'(0)$ is "small enough" so as to guarantee an interior solution). Let τ^*, A^*, and τ^{**} be defined by

$$I'(\tau^*) = R,$$

$$[p_H + \tau^*]\left[R - \frac{B}{\Delta p}\right] = I(\tau^*) - A^*,$$

$$I'(\tau^{**}) = R - \frac{B}{\Delta p}.$$

Can the borrower raise funds? If so, what is the equilibrium level τ of "quality of investment"?

(ii) Suppose now that there are two firms (that is, two borrowers) competing on this product market. If only firm i succeeds in its project, its income is (as in question (i)), equal to R (and firm j's income is 0). If the two firms succeed (both get hold of "the technology"), they compete *à la* Bertrand in the product market and get 0 each. For simplicity, assume that the lenders observe only whether the borrower's income is R or 0, rather than whether the borrower has succeeded in developing the technology (showoffs: you can discuss what would happen if the lenders observed "success/failure"!).

So, if $q_i \equiv p_i + \tau_i$ denotes the probability that firm i develops the technology (with $p_i = p_H$ or p_L), the probability that firm i makes R is $q_i(1 - q_j)$. (This assumes implicitly that projects are independent.)

Consider the following timing. (1) Each borrower simultaneously and secretly arranges financing (if feasible). A borrower's leverage (or quality of investment) is *not* observed by the other borrower. (2) Borrowers choose whether to work or shirk. (3) Projects succeed or fail.

- Let $\hat{\tau}$ be defined by

$$I'(\hat{\tau}) = [1 - (p_H + \hat{\tau})]R.$$

  Interpret $\hat{\tau}$.
- Suppose that the two borrowers have the same initial net worth A. Find the lower bound $\hat{A}$ on A such that $(\hat{\tau}, \hat{\tau})$ is the (symmetric) Nash outcome.
- Derive a sufficient condition on A under which it is an equilibrium for a single firm to raise funds.

(iii) Consider the set up of question (ii), except that borrower 1 moves first and publicly chooses τ_1. Borrower 2 may then try to raise funds (one will assume either that τ_2 is secret or that borrower 1 is rewarded on the basis of her success/failure performance; this is in order to avoid strategic choices by borrower 2 that would try to induce borrower 1 to shirk). Suppose that each has net worth $\tilde{A}$ given by

$$\tilde{q}\left[(1 - \tilde{q})R - \frac{B}{\Delta p}\right] = I(\tilde{q} - p_H) - \tilde{A},$$

where $\tilde{q}$ satisfies

$$I'(\tilde{q} - p_H) = (1 - \tilde{q})R - \frac{B}{\Delta p}.$$

- Interpret $\tilde{q}$.
- Show that it is optimal for borrower 1 to choose $\tau_1 > \tilde{q} - p_H$.

Exercise 3.8 (equity multiplier and active monitoring). (i) Derive the equity multiplier in the variable-investment model. (Reminder: the investment $I \in [0, \infty)$ yields income RI in the case of success and 0 in the case of failure. The borrower's private benefit from misbehaving is equal to BI. Misbehaving reduces the probability of success from p_H to $p_L = p_H - \Delta p$. The borrower has cash A and is protected by limited liability. Assume that $\rho_1 = p_H R > 1$, $\rho_0 = p_H(R - B/\Delta p) < 1$ and $1 > p_L R + B$. The investors' rate of time preference is equal to 0.) Show that the equity multiplier is equal to $1/(1 - \rho_0)$.

(ii) Derive the equity multiplier with active monitoring: the entrepreneur can hire a monitor, who, at private cost cI, reduces the entrepreneur's private benefit from shirking from BI to $b(c)I$, where $b(0) = B$, $b' < 0$. The monitor must be given incentives to monitor (denote by R_m his income in the case of success). The monitor wants to break even, taking into account his private monitoring cost (so, there is "no shortage of monitoring capital").

- Suppose that the entrepreneur wants to induce level of monitoring c. Write the two incentive constraints to be satisfied by R_m and R_b (where R_b is the borrower's reward in the case of success).
- What is the equity multiplier?
- Show that the entrepreneur chooses c so as to maximize

$$\max_c \left\{ \frac{\rho_1 - 1 - c}{1 - \rho_0 + (p_H/\Delta p)[b(c) + c - B]} \right\}.$$

concave private benefit). Consider ıvestment model with a concave pri- The entrepreneur obtains $B(I)$ when 0 when behaving, where $B(0) = 0$, 0 (and $B'(0)$ large, $\lim_{I\to\infty} B'(I) = B$, $B/\Delta p) < 1$).
the borrowing capacity.
:s the shadow price v of the entrepre- ı hand vary with A?

(congruence, pledgeable income, and ɜntive scheme). The credit rationing ped in this chapter assumes that the s and investors' interests are *a priori* d that incentives must be aligned by 'epreneur enough of a stake in the case

at the entrepreneur and the investors issonant preferences with probability aturally aligned interests with proba- hich prevails is unknown to both sides ıg stage and is discovered (only) by the just before the moral-hazard stage.
sely, consider the fixed-investment ion 3.2. The investors's outlay is $I - A$ and an expected rate of return equal ɜpreneur is risk neutral and protected ɔility. With probability x, interests are ɜ entrepreneur obtains private benefit ving (the probability of success is p_L) ıving (probability of success p_H). With $- x$, interests are aligned: the entre- ng her private benefit B coincides with ɔability of success p_H.
a "simple incentive scheme" in which eur receives R_b in the case of success case of failure. R_b thus measures the incentive scheme."
it may be optimal to choose a low- ntive scheme if preferences are rather low) and that the incentive scheme is gh-powered if preferences are rather ıigh).
ıt one cannot improve on simple incen- by presenting the entrepreneur with a options (two outcome-contingent in- ıes) from which she will choose once she learns whether preferences are congruent or dissonant.

Exercise 3.11 (retained-earnings benefit). An entrepreneur has at date 1 a project of fixed size with characteristics $\{I^1, R^1, p_H^1, p_L^1, B^1\}$ (see Section 3.2). This entrepreneur will at date 2 have a different fixed size project with characteristics $\{I^2, R^2, p_H^2, p_L^2, B^2\}$, which will then require new financing. So, we are considering only a short-term loan for the first project. Retained earnings from the first project can, however, be used to defray part of the investment cost of the second project. Assume that all the characteristics of the second project are known at date 1 except B^2, which is distributed on $[\underline{B}^2, \bar{B}^2]$ according to the cumulative distribution $F(B^2)$. Assume for simplicity that $\underline{B}^2 > \Delta p^2(p_H^2 R^2 - I^2)/p_H^2$. The characteristics of the second project become common knowledge at the beginning of date 2.

(i) Compute the shadow value of retained earnings. (Hint: what is the entrepreneur's gross utility in period 2?)

(ii) Show that it is possible that the first project is funded even though it would not be funded if the second project did not exist and even though the entrepreneur cannot pledge at date 1 income resulting from the second project.

Exercise 3.12 (investor risk aversion and risk premia). One of the key developments in the theory of *market* finance has been to find methods to price claims held by investors. Market finance emphasizes state-contingent pricing, the fact that 1 unit of income does not have a uniform value across states of nature. This book assumes that investors are risk neutral, and so it does not matter how the pledgeable income is spread across states of nature. This assumption is made only for the sake of computational simplicity, and can easily be relaxed.

Consider a two-date model of market finance with a representative consumer/investor. This consumer has utility of consumption $u(c_0)$ at date 0, the date at which he lends to the firm, and utility of consumption $u(c(\omega))$ at date 1, date at which he receives the return from investment. There is macroeconomic uncertainty in that the representative consumer's date-1 consumption depends on the state of nature ω. The state of nature describes both what happens

in this particular firm and in the rest of the economy (even though aggregate consumption is independent of the outcome in this particular firm to the extent that the firm is atomistic, which we will assume).

Suppose that the entrepreneur works. Let S denote the event "the project succeeds" and F the event "the project fails." Let

$$q_S = E\left[\frac{u'(c(\omega))}{u'(c_0)} \;\middle|\; \omega \in S\right]$$

and

$$q_F = E\left[\frac{u'(c(\omega))}{u'(c_0)} \;\middle|\; \omega \in F\right].$$

The firm's activity is said to covary positively with the economy (be "procyclical") if $q_S < q_F$, and negatively (be "countercyclical") if $q_F < q_S$.

Suppose that

$$p_H q_S + (1 - p_H) q_F = 1.$$

(i) Interpret this assumption.

(ii) In the fixed-investment model of Section 3.2 (and still assuming that the entrepreneur is risk neutral), derive the necessary and sufficient condition for the project to receive financing.

(iii) What is the optimal contract between the investors and the entrepreneur? Does it involve maximum punishment ($R_b = 0$) in the case of failure? How would your answer change if the entrepreneur were risk averse? (For simplicity, assume that her only claim is in the firm. She does not hold any of the market portfolio.)

Exercise 3.13 (lender market power). (i) *Fixed investment.* An entrepreneur has cash amount A and wants to invest $I > A$ into a (fixed-size) project. The project yields $R > 0$ with probability p and 0 with probability $1 - p$. The probability of success is p_H if the entrepreneur works and $p_L = p_H - \Delta p$ ($\Delta p > 0$) if she shirks. The entrepreneur obtains private benefit B if she shirks and 0 otherwise. The borrower is protected by limited liability and everyone is risk neutral. The project is worthwhile only if the entrepreneur behaves.

There is a single lender. This lender has access to funds that command an expected rate of return equal to 0 (so the lender would content himself with a 0 rate of return, but he will use his market power to obtain a superior rate of return). Assume

$$V \equiv p_H R - I > 0$$

and let $\bar{A}$ and $\hat{A}$ be defined by

$$p_H\left[R - \frac{B}{\Delta p}\right] = I - \bar{A}$$

and

$$p_H \frac{B}{\Delta p} - \hat{A} = 0.$$

Assume that $\bar{A} > 0$ and that the *lender* makes a take-it-or-leave-it offer to the borrower (i.e., the lender chooses R_b, the borrower's compensation in the case of success).

- What contract is optimal for the lender?
- Is the financing decision affected by lender market power (i.e., compared with the case of competitive lenders solved in Section 3.2)?
- Draw the borrower's *net* utility (i.e., net of A) as a function of A and note that it is nonmonotonic (distinguish four regions: $(-\infty, \bar{A})$, $[\bar{A}, \hat{A})$, $[\hat{A}, I)$, $[I, \infty)$). Explain.

(ii) *Variable investment.* Answer the first two bullets in question (i) (lender's optimal contract and impact of lender market power on the investment decision) in the variable-investment version. In particular, show that lender market power reduces the scale of investment. (Reminder: I is chosen in $[0, \infty)$. The project yields RI if successful and 0 if it fails. Shirking, which reduces the probability of success from p_H to p_L, yields private benefit BI. Assume that $p_H R > 1 > p_H(R - B/\Delta p)$. Hint: show that the two constraints in the lender's program are binding.)

Exercise 3.14 (liquidation incentives). This exercise extends the fixed-investment model of Section 3.2 by adding a signal on the profitability of the project that (a) accrues after effort has been chosen, and (b) is privately observed. (The following model is used as a building block in a broader context by Dessi (2005).)

An entrepreneur has cash A and wants to invest $I > A$ into a project. The project yields R (success) or 0 (failure) at the end. An intermediate signal reveals the probability y that the project will succeed, with $y = \bar{y}$ or $\underline{y}$ ($\bar{y} = \underline{y} + \Delta y$ and $\Delta y > 0$). The probability, p, that $y = \bar{y}$ depends on the entrepreneur's effort. If the entrepreneur behaves, then $p = p_H$ and the entrepreneur receives no private benefit. If the entrepreneur misbehaves, then $p = p_L$