3. Outside Financing Capacity

Entrepreneur benefits from signing a "random financing contract," though.
(i) Consider a contract in which the borrower invests \( A \in [0, A] \) of her own money, the project is financed with probability \( x \), and the borrower receives \( R_0 \) in the case of success and 0 otherwise. Write the investors’ breakeven condition.
(ii) Show that (provided the NPV, \( p_tR - I \), is positive) it is optimal for the borrower to invest \( A = A^* \).

How does the probability that the project is undertaken vary with \( A^* \)?

Exercise 3.2 (impact of entrepreneurial risk aversion). Consider the fixed-investment model developed in this chapter: an entrepreneur has cash amount \( A \) and wants to invest \( I > A \) into a project. The project yields \( R > 0 \) with probability \( p \) and \( 1 - p \) otherwise. The probability of success is \( p_t \) if the entrepreneur works and \( p_{t - 1} - \Delta p \) if she shirks. The entrepreneur obtains private benefit \( B^* \) if she shirks and 0 otherwise. Assume that
\[
I > p_t(R - B^*) - \frac{B^*}{\Delta p}.
\]
(Suppose that \( p_tR + B^* < I \), so the project is not financed if the entrepreneur shirks.)

(i) In contrast with the risk-neutrality assumption of this chapter, assume that the entrepreneur has utility for consumption:
\[
u(c) = \begin{cases} 
c & \text{if } c \geq c_0, 
-\infty & \text{otherwise.}
\end{cases}
\]

Assume that \( A \geq A_0 \) to ensure that the entrepreneur is not in the "\(-\infty\) range" in the absence of financing.

Compute the minimum equity level \( A^* \) for which the project is financed by risk-neutral investors when the market rate of interest is 0. Discuss the difference between \( p_t = 1 \) and \( p_t < 1 \).

(ii) Generalize the analysis to risk aversion. Let \( u(c) \) denote the entrepreneur’s utility from consumption with \( u^* > 0, u^* < 0 \). Conduct the analysis assuming either limited liability or the absence of limited liability.

Exercise 3.3 (random private benefits). Consider the variable-investment model: an entrepreneur initi-

3.9 Exercises

Exercise 3.1 (random financing). Consider the fixed-investment model of Section 3.2. We know that if \( A \geq A^* \),
\[
I > p_t(R - B^*) - \frac{B^*}{\Delta p}.
\]
It is both optimal and feasible for the borrower to sign a contract in which the project is undertaken for certain. We also noted that for \( A < A^* \), the borrower cannot convince investors to undertake the project with probability 1. With \( A > 0 \), the entre-

private benefit \( B^* \) from shirking and \( C^* \) from working and shirking.

The return for a firm is
\[
R = M - B^* + \frac{B^*}{\Delta p} \leq I
\]
where \( M > D > 0 \).

Assume that \( p_t(M - B^*) < I \), Nash equilibrium in contracts (where the entrepreneur negotiates with investors, but he can not predict whether the other obtains funding). In a first step, assume that his project’s risks or technology, so that nothing is learned from failure of the other firm concerning the borrower.

(i) Show that there is a cutoff \( A^* \) such that the entrepreneur \( i \) obtains no funding. (ii) Show that there is a cutoff \( A^* \) such that the entrepreneur \( i \) for \( i = 1, 2, \) both firms receive funds.

(iii) Show that if \( A < A^* \), \( A^* \) for \( i \) exist two (pure-strategy) equilibria.

(iv) The previous questions ha when investment projects are independent market competition makes it more difficult to obtain financing. Let that when projects are correlated, competition may facilitate financier financials to benchmark the entrepreneur's on that of competing firms.

Let us change the entrepreneur slightly:
\[
u(c) = \begin{cases} 
c & \text{if } c \geq c_0, 
-\infty & \text{otherwise.}
\end{cases}
\]
That is, the entrepreneur is infinite and low \( c_0 \) (this assumption is stronger than it simplifies the computation).

Suppose, first, that only one firm \( c \) that the necessary and sufficient \( c \) investment to take place is
\[
p_t(M - B^*) - c_0 \geq I
\]
3. Outside Financing Capacity

g on from question (iv), suppose now wo firms and that their technologies are unrelated in that if both invest and earn, then they both succeed or they technically oriented reader, there is no lying state variable or distribution unli common to both firms such that succeeds if \( w < \bar{p}_1 \), always fails if succeeds and only if the entrepreneur \( \bar{p}_2 < w < \bar{p}_1 \).

Gullible for both entrepreneurs: Conclude that product-market com- ecitate financing.

ontious investment and decreases.

Consider the continuous-
de-ali, with one modification: invest to return \( R(\tau) \) in the case of success, \( 0 \) in the case of failure, where \( R^\tau > 0, R^\tau < 0, \tau = 0 > 1 \). The rest of the model. The entrepreneur starts with cash \( \bar{A} \) of success is \( \bar{p}_1 \) if the entrepreneur succeeds, \( -\bar{p}_2 - \Delta \bar{p} \) if she misbehaves. The brains private benefit \( B \) if she misbehaves. Only the final outcome is \( I(\tau) \) define the level of investment: total surplus: \( p_R R(\tau) \).

Investment \( I(\tau) \) vary with assets? the shadow value of assets (the borrower’s gross utility with respect to the level of assets?

negotiation and debt forgiveness. \( \bar{p} \) the multiplier \( \bar{p} \) given by equation: assumed that it is optimal to spec- the borrower large enough that the rain (\( \bar{C}_0 \)) is satisfied. Because condi- that the project has negative NPV misuse, such a specification is when the contract cannot be renegoti- e case of this exercise is to check in a way that the borrower cannot gain an agreement in which \( \bar{C}_0 \) is not which is potentially renegotiated be- r chooses her effort. While there is way to prove this result, some insights

are gleaned from this pedestrian approach. Indeed, the exercise provides conditions under which the lender is willing to forgive debt in order to boost incentives (the analysis will bear some resemblance to that of liquidity shocks in Chapter 5, except that the lender’s concession takes the form of debt forgiveness rather than cash infusion).

(i) Consider a loan agreement specifying investment \( I \) and stake \( R_0 < B(\Delta p) \) for the borrower. Suppose that the loan agreement can be renegotiated after it is signed and the investment is sunk and before the borrower chooses her effort. Renegotiation takes place if and only if it is mutually advantageous. Show that the loan agreement is renegotiated if and only if

\[
(\Delta p)R_1 - \bar{p}_1 B(\Delta p) + \bar{p}_1 R_0 > 0.
\]

(ii) Interpret the previous condition. In particular, show that it can be obtained directly from the general theory. Hint: consider a fictitious, “fixed-investment” project with income \( (\Delta p)R(1) \), investment \( I(\tau) \), and cash on hand \( p_R p_1 R_1 \).

(iii) Assume for instance that the entrepreneur makes a take-it-or-leave-it offer in the renegotiation (that is, the entrepreneur has the bargaining power). Compute the borrowing capacity when \( R_0 < B(\Delta p) \) and the loan agreement is renegotiated.

(iv) Use a direct, rational expectations argument to point out in a different way that there is no loss of generality in assuming \( R_0 > B(\Delta p) \) (and therefore no renegotiation).

Exercise 3.7 (strategic leverage). (i) A borrower has assets \( A \) and must find financing for an investment \( I(\tau) \). As usual, the project yields \( R(\tau) \) (success) or \( 0 \) (failure). The borrower is protected by limited liability. The probability of success is \( p_1 + \tau \) or \( p_1 + \tau \), depending on whether the borrower works or shirks, with \( \Delta p = p_1 - p_1 > 0 \). There is no private benefit when working and private benefit \( B \) when shirking. The financial market is competitive and the expected rate of return demanded by investors is equal to \( 0 \). It is never optimal to give incentives to shirk.

The investment cost \( I \) is an increasing and convex function of \( \tau \) (it will be further assumed that \( p_1 R > I(\tau) ) \), that in the relevant range \( p_1 + \tau < \tau < 1 \), and that \( (\Delta p)^\tau \) is “small enough” so as to guarantee an interior solution. Let \( \tau, A^\ast, \tau^\ast \) be defined by

\[
\begin{align*}
\Gamma(\tau^\ast) &= R, \\
(\bar{p}_1 + \tau^\ast) R - B(\Delta p) &= I(\tau^\ast) - A^\ast, \\
\Gamma(\tau^\ast) &= R - B(\Delta p) + \bar{p}_1 R_0 > 0.
\end{align*}
\]

Can the borrower raise funds? If so, what is the equilibrium level \( \tau \) of “quality of investment”?

(iii) Suppose now that there are two firms (that is, two borrowers) competing on this product market. If only firm succeeds in its project, its income is in question (ii), equal to \( R(\tau) \) (and firm \( j \)'s income is \( 0 \). If the two firms succeed (both get hold of “the technology”), they compete a la Bertrand in the product market and get \( 0 \) each. For simplicity, assume that the lenders observe only whether the borrower’s income is \( R \) or \( 0 \), rather than whether the borrower has succeeded in developing the technology (showoffs: you can discuss what would happen if the lenders observed “success/failure”).

So, if \( q_I = p_I + \tau \) denotes the probability that firm \( i \) develops the technology (with \( p_I = p_1 + \tau \)), the probability that firm \( i \) makes \( R \) is \( q_I (1 - q_I) \). This assumes implicitly that projects are independent.

Consider the following timing. (1) Each borrower simultaneously and secretly arranges financing (if feasible). A borrower’s leverage (or quality of investment) is not observed by the other borrower. (2) Borrowers choose whether to work or shirk. (3) Projects succeed or fail.

Let \( \tilde{\tau} \) be defined by

\[
\Gamma(\tilde{\tau}) = (1 - (\bar{p}_1 + \tilde{\tau})) R.
\]

Interpret \( \tilde{\tau} \).

Suppose that the two borrowers have the same initial net worth \( A \). Find the lower bound \( \tilde{\tau} \) on \( A \) such that \( (\tilde{\tau}, A) \) is the (symmetric) Nash outcome.

Derive a sufficient condition on \( A \) under which it is an equilibrium for a single firm to raise funds.

(iii) Consider the set up of question (ii), except that borrower 1 moves first and publicly chooses \( \tau_1 \), borrower 2 may then try to raise funds (one will assume either that \( \tau_2 \) is secret or that borrower 1 is rewarded on the basis of her success/failure performance; this is in order to avoid strategic choices by borrower 2 that would try to induce borrower 1 to shirk). Suppose that each has net worth \( \bar{A} \) given by

\[
\bar{A} = (1 - \bar{p}_1) R - B(\Delta p) = \bar{A} - (\bar{q} - \bar{p}_1) - \bar{A},
\]

where \( \bar{q} \) satisfies

\[
\Gamma(\bar{q} - \bar{p}_1) = (1 - \bar{q}) R - B(\Delta p).
\]

• Interpret \( \bar{q} \).
• Show that it is optimal for borrower 1 to choose \( \tau_1 = \bar{q} - \bar{p}_1 \).

Exercise 3.8 (equity multiplier and active monitoring). (i) Derive the equity multiplier in the variable investment model. (Reminder: the investment \( \bar{p}_1 \) in \( [0, \infty) \) yields income \( R(\tau) \) in the case of success and \( 0 \) in the case of failure. The borrower’s private ben- efit from misbehaving is equal to \( B(\Delta p) \). Misbehaving reduces the probability of success from \( p_1 + \tau \) to \( p_1 + \Delta \). The borrower has cash \( A \) and is protected by limited liability. Assume that \( p_1 = p_1 R > 1, p_1 = p_1 R(1 - B(\Delta p)) < 1 \), and \( 1 > p_1 R + B \). The in- vestors’ rate of time preference is equal to \( 0 \). Show that the equity multiplier is equal to \( 1 / (1 - \bar{p}_1) \).

(ii) Derive the equity multiplier with active moni- toring: the entrepreneur can hire a monitor, who, at private cost \( cI \), reduces the entrepreneur’s private ben- efit from shirking from \( B(\Delta p) \) to \( B(\Delta p) - cI \). The monitor must be given incentives to monitor (denote by \( R_0 \) her income in the case of success). The monitor wants to break even, taking into account his private monitoring cost (so, there is “no shortage of monitoring capital”).

• Suppose that the entrepreneur wants to induce level of monitoring \( c(I) \). Write the two incentive constraints to be satisfied by \( R_0 \) and \( R_0 \) (where \( R_0 \) is the borrower’s reward in the case of suc- cess).

• What is the equity multiplier?

• Show that the entrepreneur chooses \( c \) so as to maximize

\[
\max_c \left\{ 1 - \rho_1 - (p_1 + \bar{p}_1) \rho_1 B(c) + c - B \right\}
\]
Concave private benefit. Consider investment model with a concave private benefit. The entrepreneur obtains $B(I)$ when $0 \leq b$, where $B(0) = 0$ and $B(I) > 0$ for $0 < I < I^*$. The entrepreneur earns $I - B(I) > 0$ if he invests $I < I^*$. The borrowing capacity is $s$, the shadow price of the entrepreneur's hand with availability $V$.

Coequivalence, pledgeable income, and entail scheme. The credit rationing problem in this chapter assumes that the $s$ and investors' interests are a priori unrelated unless the entrepreneur is aligned by a stake in the state of the world and the investor's financial preferences with probability $\lambda$ is adjusted by the entrepreneur's partial interest in the project's success.

Exercise 3.11 (retained-earnings benefit). An entrepreneur has at date 1 a project of fixed size with characteristics $(I^*, B^*, p_0, p_*$) (see Section 3.2). This entrepreneur will at date 2 have a different fixed-size project with characteristics $(I^*, B^*, p_0, p_*$), which will require new financing. So, we are considering a short-term loan for the new project. Retained earnings from the first project can, however, be used to defray part of the investment cost of the second project. Assume that all the characteristics of the second project are known at date 1 except $B^*$, which is distributed on $\mathbb{B}^*$ according to the cumulative distribution $F(B^*)$. Assume for simplicity that $B^* > \Delta p^2(p_0^2 - I^2)/p_0$. The characteristics of the second project become common knowledge at the beginning of period 2.

(i) Compute the shadow value of retained earnings. Hint: what is the entrepreneur's gross utility in period 2?

(ii) Show that it is possible that the first project is funded even though it would not have been funded if the second project did not exist and even though the entrepreneur cannot pledge at date 1 income resulting from the second project.

Exercise 3.12 (investor risk aversion and risk premium). One of the key developments in the theory of market finance has been to find methods to price claims held by investors. Market finance emphasizes state-contingent pricing, the fact that 1 unit of income does not have a uniform value across states of nature. This book assumes that investors are risk neutral and so it does not matter how the marketable income is spread across states of nature. This assumption is made only for the sake of computational simplicity, and it can be relaxed.

Consider a two-date model of market finance with a representative consumer/investor. This consumer has utility $u(c_0)$ at date 0, the date at which he lends to the firm, and utility of consumption $u(c_1)$ at date 1, date at which he receives the return from investment. There is macroeconomic uncertainty in that the representative consumer's date-1 consumption depends on the state of nature $\omega$. The state of nature describes both what happens in this particular firm and in the rest of the economy (even though aggregate consumption is independent of the outcome in this particular firm to the extent that the firm is atomistic, which we will assume).

Suppose that the entrepreneur works. Let $\delta$ denote the event the project succeeds and $\delta$ the event the project fails. Let

$$q_1 = F \left( \frac{u'((c_0))}{u'(c_1)} \right) \quad \omega \in \bar{Y}$$

and

$$q_2 = F \left( \frac{u'((c_0))}{u'(c_1)} \right) \quad \omega \in \bar{F}.$$

The firm's activity is said to covary positively with the economy (be "procyctical") if $q_1 < q_2$, and negatively (be "countercyctical") if $q_1 > q_2$.

Suppose that

$$p_0 q_1 + (1 - p_0) q_2 = 1.$$

(i) Interpret this assumption.

(ii) In the fixed-investment model of Section 3.2 and still assuming that the entrepreneur is risk neutral, derive the necessary and sufficient condition for the project to receive financing.

(iii) What is the optimal contract between the investor and the entrepreneur? Does it involve maximum punishment ($E - 0$) in the case of failure? How would your answer change if the entrepreneur were risk averse? (For simplicity, assume that her only claim is in the firm. She does not hold any of the market portfolio.)

Exercise 3.13 (lender market power). (i) Fixed investment. An entrepreneur has cash amount $A$ and wants to invest $I > A$ into a fixed-size project. The project yields $R > 0$ with probability $p$ and $0$ with probability $1 - p$. The probability of success is $p_0$ if the entrepreneur works and $p_1 = p_0 - \Delta p$ ($\Delta > 0$) if she shirks. The entrepreneur obtains private benefit $B$ if she shirks and 0 otherwise. The borrower is protected by limited liability and everyone is risk neutral. The project is worthwhile only if the entrepreneur behaves.

There is a single lender. The lender has access to funds that command an expected rate of return equal to 0 (so the lender would content himself with a 0 rate of return, but he will use his market power to obtain a superior rate of return). Assume

$$V = p_0 R - I > 0$$

and let $\lambda$ and $\lambda$ be defined by

$$p_0 \left( R - \frac{B}{\Delta p} \right) - I - \lambda = 0$$

and

$$p_1 \frac{B}{\Delta p} - \lambda = 0.$$

Assume that $A > 0$ and that the lender makes a take-it-or-leave-it offer to the borrower (i.e., the lender chooses $R$, the borrower's compensation in the case of success).

• What contract is optimal for the lender?

• Is the financing decision affected by lender market power (i.e., compared with the case of competitive lenders solved in Section 3.2)?

• Draw the borrower's net utility (i.e., net of $A$) as a function of $A$ and note that it is monotonic (distinguish four regions: $(-\infty, \lambda), (\lambda, \bar{\lambda}, (\bar{\lambda}, \bar{\lambda}, (\bar{\lambda}, \infty)$). Explain.

(ii) Variable investment. Answer the first two butts in question (i) (lender's optimal contract and impact of lender market power on the investment decision) in the variable-investment version. In particular, show that lender market power reduces the scale of investment. (Reminder: $\lambda$ is chosen in $[0, \infty$.) The project yields $R(I)$ if successful and 0 if it fails. Shifting, which reduces the probability of success from $p_0$ to $p_1$, yields private benefit $B$. Assume that $p_0 R > 1 > p_0 (R - B/\Delta p)$. Hint: show that the two constraints in the lender's program are binding.)

Exercise 3.14 (liquidity incentives). This exercise extends the fixed-investment model of Section 3.2 by adding a signal on the profitability of the project that (a) accrues after effort (or after a choice of $A$), and (b) is privately observed. (The following model is used as a building block in a broader context by Dessi (2005).)

An entrepreneur has cash $A$ and wants to invest $I > A$ into a project. The project yields $R$ (success) or 0 (failure) at the end. An intermediate signal reveals the probability $\gamma$ that the project will succeed, with $\gamma = \gamma$ or $\gamma = \gamma + \Delta$ and $\gamma > 0$. The probability, $p$, that $\gamma$ depends on the entrepreneur's effort. If the entrepreneur behaves, then $p = p_0$ and the entrepreneur receives no private benefit. If the entrepreneur misbehaves, then $p = p_1.$