Issue Linkage and The Enforcement of International Agreements*

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Abstract

What is the optimal design for a set of self-enforcing international agreements? We study a dynamic model with two asymmetric countries and n policy dilemmas, and show that having separate agreements on different policy issues is generally suboptimal. The outcome of separate agreements can always be replicated by one agreement linking all issues at stake, while issue linkage may improve the best enforceable outcome by: a) improving the allocation of scarce enforcement power among issues/countries; and b) creating additional enforcement power when policy issues are substitutes (countries' objective functions are submodular). Credible sanctions are outlined that are robust to WTO-induced renegotiation.

Keywords: Issue linkage; International agreements; Policy cooperation; Cross-border spillovers; International institutions; Retaliation; Renegotiation; Sanctions; WTO; Dispute settlement understanding;

J.E.L. classification: E61; F13; F42; H77

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1 Introduction

Globalization is making nations more interdependent than ever before. Higher interdependence means greater externalities of domestic policies imposed on neighbor countries. Greater cross-border spillovers imply an increased need for international policy cooperation. It is well known that the sovereignty of nations requires international agreements to be “self-enforcing,” that is constructed so that each country prefers to respect their requirements in the absence of an international authority able to enforce them. In this paper we address the following questions: What is the optimal design for a set of self-enforcing international policy agreements? To which degree should different policy issues be linked in international agreements? Are institutions limiting nations’ ability to link more policy issues in the same agreement welfare-enhancing?

The “real world” of international relations appears dominated by the belief that keeping international agreements on different policy issues separate is the best route to international cooperation. WTO rules explicitly forbid using trade sanctions to discipline cooperation on policy issues outside WTO, and even discourage cross-retaliation between different trade areas (Article 22.3). Not only do we observe separate international agreements on trade, monetary, environmental, and defense policies; we even have separate agreements on, say, the control of CFCs emissions and the protection of whales.

This paper shows that keeping agreements on different policy issues separate is generally not the best strategy to enforce international cooperation. The model we analyze is rather general: we consider two possibly asymmetric countries facing repeatedly \( n \) potentially interdependent policy dilemmas.\(^1\) We derive a general result showing that linking more policy issues in a single agreement is always weakly optimal from an enforcement perspective, and characterize the conditions under which it is strictly optimal. Issue linkage facilitates enforcement by improving the allocation of scarce enforcement power across issues and countries.\(^2\) When two or more policy issues are substitutes, i.e. when a country’s objective function is submodular in the outcomes of these policy dilemmas, issue linkage also increases the amount of enforcement power available. Issue linkage forces punishments and optimal violations to be simultaneous on all the linked issues. When some issues are substitutes this makes punishments harder and violations less valuable relative to the case of separate agreements, creating additional enforcement power. The converse argument does not apply when issues are complement, as a simultaneous deviation is relatively more profitable but it must be deterred whether or not issues are linked. Therefore, a “grand international agreement” on all policy issues at stake is generally optimal from an enforcement perspective: it maximizes available enforcement power and optimizes its allocation. The paper also discusses how sanctions

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\(^1\) Policy issues in need of international cooperation have strategic features similar to a Prisoner’s Dilemma. Because countries face such policy dilemmas repeatedly in time, self-enforcing international agreements can be analyzed as equilibria of infinitely repeated Prisoner’s Dilemma games.

\(^2\) For “facilitates cooperation” we will mean throughout the paper “makes countries’ incentive constraints for respecting any agreement less stringent,” so that a larger set of agreements becomes sustainable at any given intertemporal discount factor, and any given set of agreements becomes supportable at lower discount factors.
supporting international agreements should be constructed when international institutions exist that induce renegotiations after agreements’ violations. We show that crucial to an effective enforcement of international agreements is that sanctions are beneficial to countries that exert them, and that a country that violates an agreement must at least compensate other countries for the damage it caused them. We argue that the current WTO’s Dispute Settlement Understanding should be revised in this direction.

From a formal point of view our model builds upon, generalizes and extends Bernheim and Whinston’s (1990) and Spagnolo’s (1999) models of collusion with multistate contact, and van Damme’s (1989) analysis of renegotiation-proof equilibria in Prisoner’s Dilemma supergames. From an economic point of view, repeated games have been used before by many authors to model policy cooperation among sovereign countries: for example by Bagwell and Staiger (1990), Hungerford (1991), Riezman (1991), and Maggi (1999) for free trade agreements (see Staiger 1995 for an excellent overview); by Canzoneri and Henderson (1991) and Currie and Levine (1993) for monetary policy cooperation; by Barret (1994) for international environmental agreements; and by Ed-erinton (2001, 2002) to analyze linkages between two policy instruments, an efficient and an inefficient one (trade taxes and domestic labor/environmental standard), both of which affect one policy issue, trade openness. All these papers focus on a single policy issue/dilemma, they do not tackle the questions addressed here. Early contributions on “issue linkage” in international environmental agreements, for example by Folmer, van Mouche and Ragland (1993), Carraro and Siniscalco (1995, 1997), and Cesar and de Zeeuw (1996), and the more recent work of Abrego et al. (2001) and Conconi and Perroni (2002), do focus on multiple international policy issues/dilemmas. However, these papers address static strategic situations or international negotiations, not the enforcement of international agreements.

The model is presented in Section 2; Sections 3 and 4 analyze issue linkage; Section 5 discusses renegotiation and sanctions; and Section 5 briefly concludes. Proofs are in the Appendix.

2 The Model

We first we describe the static structure of the model (the policy “stage game”), then we proceed with the dynamic model and define what we mean by isolation and linkage of policy issues in international agreements.

The n-issue policy dilemma. Consider a world with two possibly asymmetric countries indexed by the subscript $J = A, B$.

Countries are assumed to be individual, rational players, and we will use the terms “country” and “government” as synonyms. In all what follows we abstract from any potential “transaction,” “bureaucratic,” or “complexity” cost of issue linkage.

The two countries make simultaneously a policy choice or action $p_J = (p_{J1}, \ldots, p_{Jn}) \in \mathbb{R}^n$.

3At the cost of increased complexity the model can be extended to more than two countries along the lines of Maggi (1999)); we see no reason why our qualitative results should change.
**Policy Profile**

\[ p = (p_A, p_B) = ((p_{A1}, ..., p_{An}), (p_{B1}, ..., p_{Bn})) \in P = P_A \times P_B \]

specifies for each country a policy choice with respect to each of \( i \in N = \{1, ..., n\} \) policy issues. For example, we may think of tariffs on various traded goods, environmental policies, monetary policy, defence policy and so on. Let \( u_J (p) \) denote the static payoff function for country \( J \). For each policy issue and each country we consider two salient modes of behavior, \( C \) for cooperation and \( D \) for defection (selfish policy action). The corresponding stage game with strategy space \( P_J = P_{J1} \times \cdots \times P_{Jn} = \{C, D\}^n \) is denoted by \( \Gamma \).

For policy profile \( p \) and policy issue \( i \) we denote by \( \Gamma_i (p) \) the \( 2 \times 2 \)–game where both countries can cooperate or defect only with respect to policy issue \( i \) and play policy profile \( p \) otherwise, that is

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with

\[
\begin{align*}
c_{Ai} (p) &= u_J (((C, p_{J,-i}), (C, p_{J,-i})), \\
d_{Ai} (p) &= u_J (((D, p_{J,-i}), (D, p_{J,-i})), \\
b_{Ai} (p) &= u_J (((D, p_{J,-i}), (C, p_{J,-i})), \\
a_{Ai} (p) &= u_J (((C, p_{J,-i}), (D, p_{J,-i}))).
\end{align*}
\]

The policy dilemma has the structure of a multi-dimensional Prisoner’s Dilemma, in that

\[
\begin{align*}
b_{Ai} (p) &> c_{Ai} (p) > d_{Ai} (p) > a_{Ai} (p) \quad \text{and} \\
c_{Ai} (p) + c_{Bi} (p) &> \max \{a_{Ai} (p) + b_{Bi} (p), a_{Bi} (p) + b_{Ai} (p)\}.
\end{align*}
\]

As in the one-dimensional Prisoner’s Dilemma, total defection \( \omega := ((D, ..., D), (D, ..., D)) \in P \) is the unique stage game equilibrium, while the second inequality guarantees that total cooperation \( \varphi := ((C, ..., C), (C, ..., C)) \in P \) is the most efficient mode of behavior.

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4 One could allow for more general (e.g. continuous) policy action sets assuming that for each country \( J \) and policy issue \( i \) there exists a selfish policy choice \( p^p_{Ji} = \arg \max_{p_{Ji}} u_J (p) \) and a cooperative policy choice \( p^C_{Ji} = \arg \max_{p_{Ji}} (u_A (p) + u_B (p)) \neq p^D_{Ji} \) that do not depend the remainder of policy profile \( p \).

5 We use the usual notational convention \( -J \) and \( p_{J,-i} \) to denote “the other player” and the “remaining actions except \( i \) in the policy action vector”.

4

4

Policy dynamics. Consider now the discounted infinite repetition \( \Gamma (\delta_A, \delta_B) \) of the n-issue policy dilemma with discount factors \( \delta_A, \delta_B < 1 \) in discrete time \( t = 1, 2, \ldots \). By an international agreement we mean a written set of rules that prescribe the correct behavior of participating countries with respect to some policy issue (or set of policy issues) together with another set of rules that specify how to proceed if a country fails to follow the first set of rules. This second set of rules will be called punishments (or sanctions). The whole set of international agreements will be the main object of interest, and will be called international order or agreement pattern. In our model an international order or agreement pattern \( \alpha = (\alpha_A, \alpha_B) \) is a strategy profile in \( \Gamma (\delta_A, \delta_B) \).

Denote by \( p^1_{\alpha}, p^2_{\alpha}, \ldots \) the path of policy profiles induced by agreement pattern \( \alpha \) if no country violates (defects from) an international agreement. Country \( J \)'s objective function is given by

\[
U_J (\alpha) = \sum_{t=1}^{\infty} \delta^t_J u_J (p^t_{\alpha}).
\]

An agreement pattern \( \alpha \) is called sustainable if no country has an incentive to violate unilaterally any agreement, i.e. if it is an equilibrium of the repeated game \( \Gamma (\delta_A, \delta_B) \). We restrict our analysis to sets of policy issues for which in principle cooperation could be achieved. More precisely, from here we suppose that \( N \) is a set of policy issues for which there exists a sustainable agreement pattern that can yield indefinite cooperation in all policy issues \( i \in N \) and thereby is efficient.

For \( n > 1 \) (more than one policy issue) the design of agreement patterns has an additional degree of freedom. Besides variations in the punishment over time, countries can vary the subset of policy issues being used for punishment. Conversely, they could exclude policy issues from being used for punishment (for example nuclear warfare). Most significantly, they can relate the set of policy issues being used for punishment to the set of policy issues on which deviations happened in the past.

Issue linkage. Let \( \alpha = (\alpha_A, \alpha_B) \) be an international order prescribing cooperative behavior with respect to all policy issues. Suppose, in period \( t \) country \( J \) breaks some agreements, i.e. violates the rules specified by \( \alpha \) with respect to a subset \( L \subset N \) of policy issues. Agreement pattern \( \alpha \) prescribes how to proceed in this case. Denote by \( P_{\alpha} (J, L) \) the set of policy issues that are used for punishment and conversely by \( C_{\alpha} (J, L) = N \setminus P_{\alpha} (J, L) \) the set of policy issues that is unaffected from this event, i.e. those where both countries continue to behave cooperatively as they did before country \( J \) defected. The event \( (J, L) \) that country \( J \) defects on the set of policy issues \( L \) will

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6 Games repeated a finite but uncertain number of periods lead to identical results for risk-neutral players since a constant probability to end the supergame can be incorporated in the discount factor.

7 A strategy profile specifies for each country and each history of previous events what to do.

8 Alternatively we could restrict attention to an "efficient cooperation subset", that is a subset of policy issues such that there exists a sustainable agreement pattern yielding indefinite cooperation on these subsets and that there is no other such subset that would induce Pareto-better equilibrium payoffs.

9 The fact that \( P_{\alpha} (J, L) \) and \( C_{\alpha} (J, L) \) have no index \( t \) implicitly contains the assumption that from here we restrict our analysis to agreement patterns \( \alpha \) with stationary punishment, i.e. for which the punishment design does not depend on the defection period \( t \).
be called a violation. Now we can introduce the main concepts.

**Definition 1** A subset of policy issues \( K \subset N \) on which countries cooperate in agreement pattern \( \alpha \) is called linked with respect to agreement \( \alpha \) if for all violations \((J, L)\) on at least one of the policy issues in \( K \) (i.e. \( L \cap K \neq \emptyset \)) countries use at least all policy issues in \( K \) for punishment, that is \( K \subset P_\alpha (J, L) \).

**Definition 2** A subset of policy issues \( K \subset N \) is called isolated in agreement pattern \( \alpha \) if (i) a violation \((J, L)\) with \( L \subset K \) is only punished within \( K \), that is \( P_\alpha (J, L) \subset K \); (ii) for a violation \((J, L)\) outside \( K \) with \( L \cap K = \emptyset \) there will be no punishment within the set \( K \) of policy issues, that is \( P_\alpha (J, L) \cap K = \emptyset \).

Note that these definitions do not depend on agreement \( \alpha \) being sustainable.\(^{10}\) Also, so far we only assumed that punishments are stationary but made no further assumptions, for example, on their severity and duration. This part of the agreement pattern is what we mean by the design of "international agreements" and will be the main object of our analysis.

**Costs and benefits of defecting.** Consider a violation \((J, L)\) of agreement pattern \( \alpha \). Denote by \( p(J, L) \) the policy profile in the violation period, i.e. \( p_{Ji} = D \) for country \( J \) and \( i \in L \) and \( C \) otherwise. To investigate the incentive for a violation \((J, L)\) with respect to agreement pattern \( \alpha \) let

\[
\begin{align*}
    b_\alpha (J, L) &= u_J (p(J, L)) - u_J (\varphi), \quad \text{and} \\
    c_\alpha (J, L) &= \delta_J \left[ \frac{u_J (\varphi)}{1 - \delta_J} - V_\alpha (J, L) \right]
\end{align*}
\]

denote respectively the benefit and the cost of this violation for the violating country \( J \), where \( V_\alpha (J, L) \) is the continuation payoff to the violating country according to agreement pattern \( \alpha \). The short term benefit \( b_\alpha (J, L) \) to violate does not depend on agreement pattern \( \alpha \) as long as \( \alpha \) prescribes cooperation with respect to all policy issues. Therefore we can omit index \( \alpha \) and write \( b(J, L) \). In contrast, the continuation payoff \( V_\alpha (J, L) \) depends on agreement pattern \( \alpha \). For example, \( \alpha \) could prescribe that the defecting country must allow the other country to defect but meanwhile cooperate itself for a number of periods before returning to mutual cooperation with respect to one policy issue. With respect to another policy issue \( \alpha \) could specify the same procedure but with a different number of periods. By doing so the defecting country could be punished and restitute the damage of its violation. For an agreement pattern \( \alpha \) to be sustainable it must be \( b(J, L) \leq c_\alpha (J, L) \) for all violations \((J, L)\). This is equivalent to

\[
    \delta_J \geq \frac{u_J (p(J, L)) - u_J (\varphi)}{U_J (\alpha) - V_\alpha (J, L)}.
\]

\(^{10}\) Clearly linked and isolated sets of policy issues could be formulated individually for each country. All results can be translated easily into this language of "individual linking" which would be more general. We prefer not to do it, however, to avoid tedious case distinctions and to simplify the presentation.
Let $\delta_J (L)$ denote country $J$’s discount factor for which this country would be indifferent between violation $(J, L)$ and not violating agreement pattern $\alpha$, i.e. $\delta_J (L) (U_J (\alpha) - V_\alpha (J, L)) = u_J (p (J, L)) - u_J (\varphi)$. Define

$$\delta_J (\alpha) := \max_{(J, L)} \delta_J (L)$$

as the smallest upper bound on discount factors for which there is no incentive to violate in agreement pattern $\alpha$ for country $J$. If for a country $\delta_J < \delta_J (\alpha)$, this country has an incentive to violate agreement pattern $\alpha$, and $\alpha$ is not sustainable. Hence $\delta_J \geq \delta_J (\alpha)$ for both countries $J = A, B$ is a necessary condition for agreement pattern $\alpha$ to be sustainable. From the “one-deviation property” for repeated games (e.g. Fudenberg and Tirole, Ch. 5) we know that this condition is also sufficient.

**Lemma 1** An agreement pattern $\alpha$ is sustainable if and only if $\delta_J \geq \delta_J (\alpha)$ for both countries $J = A, B$.

### 3 Linking Policy Issues

We now study the ”optimal design” of international agreements with respect to the question to which degree policy issues should be linked in order to sustain cooperative behavior. Our method will be to compare agreement patterns that only differ in their ”degree of linkage” but are identical otherwise.

Consider an agreement pattern $\alpha (K, M)$ with the following properties: (i) $\alpha (K, M)$ has two isolated sets $K, M$ with $K \cap M = \emptyset$ (disjoint isolated sets of policy issues) (ii) whenever a policy issue $i \in K \cup M$ is used to punish a violation the punishment pattern with respect to this issue $i$ will be the same.\(^{11}\) Define the linked version $\alpha^l (K, M)$ as another agreement pattern with the following properties. A violation $(J, L)$ with respect to policy issues in $K$ or $M$ (i.e. $L \subset K \cup M$) in agreement pattern $\alpha^l (K, M)$ is punished with respect to all policy issues in $K \cup M$ but with the same time pattern. Agreements $\alpha (K, M)$ and $\alpha^l (K, M)$ are identical otherwise (for violations outside $K \cup M$). Further, call $\alpha^*$ a fully linked agreement pattern with maximal enforcement if any violation is followed by a maximal punishment (the defector’s continuation payoff equals his maximin value) with respect to all policy issues. Of course, the grim trigger agreement where $(\omega, \omega, \ldots)$ is the continuation path after any deviation (countries quit their relationship altogether and forever) is a fully linked agreement pattern with maximal enforcement, but in general not the only one. Now we can state the main result.

**Theorem 1** (i) If the fully linked agreement pattern $\alpha^*$ with maximal enforcement is not sustainable then there exists no sustainable agreement pattern $\alpha$.

(ii) The linked version $\alpha^l (K, L)$ of a sustainable agreement pattern $\alpha (K, L)$ with isolated subsets $K, L$ of policy issues ($K \cap L = \emptyset$) is also sustainable and $\delta_J (\alpha^l (K, L)) \leq \delta_J (\alpha (K, L))$.

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\(^{11}\)If, for example, a punishment with respect to policy issue $i$ prescribe for both players to defect for two periods, then resume cooperation forever, this pattern should not depend on violation $(J, L)$. However, whether policy issue $i$ is used for punishment can depend on the violation.
The theorem tells us that in our setting ”more linking” of policy issues never destroys sustainability. The intuition is that by linking policy issues the punishments become more severe for some defections, while gains from defection do not increase. On the other hand, a non-sustainable pure agreement $\alpha$ can be made sustainable by linking policy issues. Claim (ii) of the theorem, together with Lemma 1 immediately imply a necessary and sufficient condition (characterization) when this is the case for agreement patterns with isolated subsets of policy issues.

**Corollary 1** Let $\alpha^l(K, L)$ be the linked version of agreement pattern $\alpha(K, L)$ with isolated subsets of policy issues $K, L$ such that

$$\delta_J(\alpha^l(K, L)) \leq \delta_J < \delta_J(\alpha(K, L))$$

for some country $J$ and $\delta_J(\alpha^l(K, L)) \leq \delta_J$ for both countries $J = A, B$. Then $\alpha(K, L)$ is not sustainable but the linked version $\alpha^l(K, L)$ is sustainable.

To clarify the mechanisms behind these results, in the next section we construct specific examples where a non-sustainable agreement pattern $\alpha$ becomes sustainable after a linkage of policy issues.

## 4 Gains From Issue Linkage

We mentioned that linking issues in a single agreement may both improve the allocation and increase the amount of available enforcement power, intended as the difference between short run gains from violating agreements and long run losses from being punished for that. The next two subsections clarify how this may occur.

### 4.1 Optimal reallocation of enforcement power

Linking policy agreements on different issues may foster international cooperation by ”improving the allocation of available enforcement power.” This argument was developed by Bernheim and Whinston (1990) for the analysis of collusion in more than one joint market, and is a special case within this theory. It can be illustrated with simple additively separable payoff functions given by

$$u_J(p) = u_{J1}(p_{J1}, p_{-J1}) + ... + u_{Jn}(p_{Jn}, p_{-Jn}).$$

Under this specification the policy stage game $\Gamma$ can be separated into $i = 1, ..., n$ independent policy games with payoff functions $u_{Ji}$. In this case the policy stage game is representable by $n$ possibly different $2 \times 2$—games (all with the Prisoner’s Dilemma structure)

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with the interpretation that in each period players play \( n \) policy stage games simultaneously (one for each policy issue) and maximize the discounted sum of all payoffs.

To study issue linkage consider, say, \( \alpha^l (1, 2) \) the linked version of agreement pattern \( \alpha (1, 2) \) with isolated policy issues \( \{1\}, \{2\} \). We know from Corollary 1 that if

\[
\delta_J (\alpha^l (1, 2)) < \delta_J (\alpha (1, 2))
\]

for some country \( J \) then for any \( \delta_J \) between \( \delta_J (\alpha^l (1, 2)) \) and \( \delta_J (\alpha (1, 2)) \) agreement pattern \( \alpha (1, 2) \) is not sustainable but the linked version \( \alpha^l (1, 2) \) is sustainable. To see that this condition can easily be satisfied, specify punishments to be optimal within isolated sets (i.e. a defector obtains maximin continuation utility). Note that for two policy issues there are only 3 simple defections, to defect on each one or both policy issues. The reader may want to verify that there exist payoff parameters such the property \( \delta_J (\alpha^l (1, 2)) < \delta_J (\alpha (1, 2)) \) is satisfied and contains relevant cases.\(^{12}\)

Intuitively, any asymmetry between at least two policy issues for at least one country creates scope for improving enforcement through linkages. For example, a country may find it optimal to respect an agreement on trade policy (issue 1), but not one on environmental policy (issue 2). When \( \delta_J (\alpha^l (1, 2)) < \delta_J (\alpha (1, 2)) \) a linkage of these two policy issues in a single agreement would ensure that the country finds convenient to cooperate on both issues, given that it cannot afford to be punished on trade, to the benefit of both countries. When one country is willing to cooperate a set of issues \( A \) and the other on a set \( B \), a linkage between sets \( A \) and \( B \) can be seen as an efficient “exchange” between countries (I cooperate on \( B \) in exchange for your cooperation on \( A \), and both of us realize gains from trade).

### 4.2 Creation of additional enforcement power

Let us stick to the assumption \( u_J (p) = v_J (u_{J1} (p_{A1}, p_{B1}), ..., u_{Jn} (p_{An}, p_{Bn})) \), where \( v_J (\cdot) \) is twice differentiable, but drop additive separability. When policy issues are not separable the “efficient allocation of available enforcement power” is not the only concern. Then the design of international agreements may also affect the amount of available enforcement power. To illustrate this point we can exploit the analogy with Spagnolo (1999), where the multimarket contact model is extended to the case where players’ objective functions are submodular in payoffs from different supersgames.\(^{13}\)

**Definition 3** Policy issues \( i \) and \( k \) are called substitutes for country \( J \) iff \( \frac{\partial^2 v_J (\cdot)}{\partial u_{Ji} \partial u_{Jk}} < 0 \), and complements iff \( \frac{\partial^2 v_J (\cdot)}{\partial u_{Ji} \partial u_{Jk}} > 0 \).

**Proposition 1** Let \( \alpha \) be an agreement pattern for which all policy issues are isolated \( P_\alpha (J, L) = L \) and all punishments are optimal on the punishment set (a defector obtains his maximin continuation payoff).

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\(^{12}\)In the appendix we provide a hint for the calculation.

\(^{13}\)For expositional simplicity we focus on substitute/complement policy issues; it is immediate to verify that Proposition 1 applies unchanged to submodular/supermodular policy issues.
(a) If two policy issues $j$ and $k$ are substitutes, then the short term incentive to violate simultaneously on both issues is below the sum of incentives to violate separately. For the respective cost of a violation the converse is true, i.e.

\[
\begin{align*}
    b(J, \{j, k\}) &< b(J, \{j\}) + b(J, \{k\}) \\
    c_\alpha(J, \{j, k\}) &> c_\alpha(J, \{j\}) + c_\alpha(J, \{k\}).
\end{align*}
\]

(b) If they are complements, the converse holds, i.e.

\[
\begin{align*}
    b(J, \{j, k\}) &> b(J, \{j\}) + b(J, \{k\}) \\
    c_\alpha(J, \{j, k\}) &< c_\alpha(J, \{j\}) + c_\alpha(J, \{k\}).
\end{align*}
\]

Our interpretation of Proposition 1(a) is that if some policy issues are substitutes, a linkage maintains the "positive allocative effects on available enforcement power" discussed in the previous example, and it also increases the amount of available enforcement power. When two or more issues are substitutes for a country, simultaneous defections on those issues are less attractive for two reasons. First, when the country is defecting on some issues it values relatively less short-run gains from simultaneous defections on additional substitute issues. Second, a simultaneous punishment on more policy issues is a relatively stronger threat, as when a country is not cooperating on an issue it values relatively more cooperation on substitute issues. The opposite happens when issues are complements. Then a simultaneous defection on more issues increases the value of short-run gains from defection, since the high payoffs from each issue increase the marginal value of additional payoffs from other issues, while the value of losses from a simultaneous punishment phase is reduced by the opposite effect.

What happens, if issues are complements? While Proposition 1 deals with isolated policy issues, the theorem implies that for linked agreement patterns the converse argument does not apply. A linkage between issues on which cooperation is sustainable cannot "reduce available enforcement power". Proposition 1(b) says that, with complement issues, the condition for a simultaneous deviation on two issues not being profitable is more stringent than the correspondent conditions when deviations and punishments are not simultaneous. But when some issues are not linked, nothing prevents a government from deviating simultaneously on (some or all of) them, so cooperation on those issues is sustainable only if the more stringent condition for simultaneous deviations is satisfied. This means that cooperation on two (or more) isolated complement issues can only be sustained when it can also be sustained when the issues are linked, and consequently that linking complement issues on which cooperation was sustainable cannot reduce available enforcement power (although it can still improve on its allocation).

In a related contribution Maggi (1999) analyzes multilateral trade cooperation in the presence of asymmetric trade relations. In a three-country model with separable trade relations he demonstrates that a multilateral approach facilitates international trade cooperation by allowing for "third-party sanctions." Concluding the paper, Maggi argues informally that a multilateral approach facilitates cooperation even more when trade diversion effects are taken into account, as these effects weaken threats in bilateral relations allowing each country to partially substitute trade with one partner with trade
with another. In this spirit, Proposition 1(a) can be easily reinterpreted as a proof that whenever policy cooperation (e.g. open trade) with a country is a substitute for policy cooperation with others, multilateral agreements facilitate the enforcement of policy cooperation by creating additional enforcement power (see Spagnolo 2001).

5 Retaliation and Renegotiation

At the beginning of Section 3 we said that the grim trigger agreement, where interrupting cooperation forever \((\omega, \omega, \ldots)\) is the continuation path after any deviation, is a fully linked agreement with maximal enforcement but it is not the only one. Since under a grim trigger agreement the punishing country obtains its minimal continuation payoff in the punishment phase, after a violation of an agreement both countries may agree to renegotiate it and switch to a less costly/severe form of punishment. The anticipation of this incentive to renegotiate could then undermine the sustainability of such an agreement. Farrel and Maskin (1989) first identified this problem, suggested criteria for a self-enforcing agreement to be considered (weakly) “renegotiation proof,” and showed that the problem applies to a wide range of environments. This problem is particularly relevant in the case of international agreements. This is because in the international arena there are institutions, such as the WTO, with the precise function of inducing/facilitating negotiations (and renegotiations) between countries. The WTO’s Dispute Settlement Understanding (DSU), for example, requires that after a violation is ascertained negotiations are undertaken before any retaliation can be initiated. For this reason, one should be skeptical of analyses of international agreements that rely on strategies like grim trigger. In this section we first address the question whether in our model there exist renegotiation proof agreements, and in particular (in light of our theorem) renegotiation proof agreements with maximal enforcement; then we discuss how a robust retaliation pattern could be designed in reality.

Let \(p_J^0 = ((C, \ldots, C), (D, \ldots D))\) denote the policy profile where country \(J\) cooperates on all policy issues and the other country \(-J\) defects on all policy issues. Then \(u_J(p_J^0)\) is the minimum payoff country \(J\) can ever get in a period and \(u_J(p_{-J}^0)\) is the maximum payoff country \(J\) can ever get. This implies

\[ u_J(p_J^0) < u_J(\omega) < u_J(\varphi) < u_J(p_{-J}^0). \]

Now let parameter \(\lambda\) be defined by \((1 - \lambda)u_J(p_J^0) + \lambda u_J(\varphi) = u_J(\omega)\). We can state the following.

**Proposition 2** If \(\delta_J \geq \lambda\) there exists a sustainable agreement pattern with approximately maximal enforcement supporting indefinite cooperation, such that there is no incentive for both countries to renegotiate after any deviation.

The statement implies that all the results of this paper apply if one requires international agreements to be robust with respect to renegotiation. The renegotiation-proof agreement pattern constructed in the proof prescribes that in the punishment phase a country that defected cooperates for a number of periods in which the other country
defects, after which bilateral cooperation is restored. The punishing country then gains during the punishment phase, which guarantees that it will oppose renegotiation towards milder punishments.

The proposition shows that in our model the maximal degree of cooperation can approximately be achieved through the earlier-mentioned strategies when these are so severe that after a violation all the surplus from the relationship goes to the country that suffered the violation. In this sense, the proposition generalizes an analogous result by van Damme (1989) for the single repeated Prisoner’s Dilemma. The extreme agreement pattern considered is theoretically interesting because it maximizes enforcement power, defining the theoretical frontier of what can be enforced if countries anticipate renegotiation, but may create perverse incentives, in the sense that it may render desirable for a country that the other country defects. If countries’ weight on the future is large enough, there will be many agreement patterns with analogous structure and less severe enforcement that are also sustainable and robust to renegotiation. What is important is that the retaliation pattern shares the following features with the one constructed in the proof:

a) a country that violates the agreement suffers a loss (this not as obvious as it sounds; WTO’s DSU currently prescribes not to punish a deviant if during negotiations this agrees to go back to legal behavior);

b) a country damaged by the violation is compensated (monetary compensation would be most efficient, although it is forbidden by the DSU);

c) a country required to punish a deviant is compensated for doing it.

All DSU’s problems, clearly spelled out in Anderson (2002), would be solved by a system of sanctions with these features. Besides being more effective and robust to renegotiations, such a retaliation pattern would certainly be regarded as more fair and coherent with the principles of national legal systems than the current one.

6 Concluding remarks

We showed that issue linkage cannot harm policy cooperation from an enforcement perspective, while it facilitates it by improving the allocation of scarce international enforcement power among policy issues and countries, and by creating additional enforcement power when some policy issues are substitutes. In a complete information world, these forces point at a single “grand agreement” that links all policy issues as the optimal arrangement from an enforcement perspective. The presence of international institutions inducing renegotiation was shown not to alter this general conclusion, provided the sanctions that discipline agreements are designed in a sensible way (which is currently not the case).

These results are a first step towards a theory of the optimal design for set of agreements in an international framework. Their robustness must be verified by incorporating other important features of reality. For example, the policy of punishing all deviations with the strongest available sanctions, optimal in this paper’s complete information framework, may not be optimal in an imperfect information world where stronger punishments are more costly because they are implemented along the equilibrium path (e.g.
Green and Porter, 1984). Analogously, if countries make mistakes, too strong punishments may be suboptimal from an *ex ante* point of view; linking all issues in one agreement might be too risky. Paralleling the literature on optimal law enforcement after Becker’s (1968) result on maximal sanctions (see Polinsky and Shavell, 2000), future work should ask when and why “fit-the-crime” punishments may do better than maximal sanctions in the enforcement of international cooperation, and if and how this may affect our conclusions on the optimal design international agreements.
7 Appendix

Description and discussion of the repeated game \( \Gamma(\delta_A, \delta_B) \). Denote by \( H^t \) the set of histories of length \( t \) (as usual \( H^0 = \emptyset \) is the empty history) and by \( H = \bigcup_{t=0}^{\infty} H^t \) the set of all histories. A strategy \( \alpha_J : H \rightarrow P_J \) of country \( J \) maps any history to an action. In this paper a strategy profile is the main object of investigation and because of its interpretation \( \alpha = (\alpha_A, \alpha_B) \) is called agreement pattern. An agreement pattern \( \alpha \) and a history \( h \in H^t \) together induce an outcome path

\[
\pi_\alpha(h) = (p^{t+1}_\alpha(h), p^{t+2}_\alpha(h), \ldots)
\]

\[
= (\alpha_A(h), \alpha_B(h)), (\alpha_A(h), \alpha_B(h)), \ldots).
\]

Denote in particular by \( \pi_\alpha := (p^1_\alpha, p^2_\alpha, \ldots) \) the outcome path induced by agreement \( \alpha \) and the empty history. The payoff functions for \( \Gamma(\delta_A, \delta_B) \) are then given by

\[
U_J(\alpha) = \sum_{t=1}^{\infty} \delta_J \sum_{\pi}(p^t_{\alpha}(h^{t}(J,L))).
\]

Another important role plays the (punishment) payoff a country obtains after it defected. We assume that this payoff is stationary (does not depend on the period \( t \) where the violation \( (J,L) \) takes place). It is discounted from the perspective of the violation period and given by

\[
V_J(J,L) = \sum_{\tau=1}^{\infty} \delta_J \sum_{\pi}(p^{t+\tau}_{\alpha}(h^{t}(J,L))).
\]

with history

\[
h^t(J,L) = \{(C,\ldots,C),(C,\ldots,C),\ldots,(C,\ldots,C),(C,\ldots,C)\).
\]

\[
\text{policy profiles } (p^1_{J,0},p^2_{J,0},\ldots,p^{t-1}_{J,0},p^{t}_{J,0})
\]

\[
\text{policy issue set } L
\]

\[
\text{policy profile } (p^t_{J,P},p^t_{J,F})
\]

Proof of Theorem 1. (i) If \( \alpha^* \) is not sustainable we know by Lemma 1 that there is a country with \( \delta_J < \delta_J(\alpha^*) \) and there exists an incentive for a violation \( (J,L) \) such that \( b(J,L) > c_{\alpha^*}(J,L) \) or

\[
\delta_J < \frac{u_J(p(J,L)) - u_J(\varphi)}{1 - \delta_J} - V_{\alpha^*}(J,L).
\]

Let conversely \( \alpha \) be a sustainable pure agreement. Their outcome paths are identical up to period \( t \) where the violation \( (J,L) \) takes place including this period. Since \( \alpha^* \) is
a fully linked pure agreement with maximal enforcement we know that for the defector \( J \) his continuation utility after the defection is minimal, hence \( V_{\alpha}^* (J, L) \leq V_{\alpha} (J, L) \). Together this implies
\[
\delta_J < \frac{u_{J} (p (J, L)) - u_{J} (\varphi)}{\frac{u_{J} (\varphi)}{1 - \delta_J} - V_{\alpha}^* (J, L)} \leq \frac{u_{J} (p (J, L)) - u_{J} (\varphi)}{\frac{u_{J} (\varphi)}{1 - \delta_J} - V_{\alpha} (J, L)}
\]
or \( b (J, L) > c_{\alpha} (J, L) \) which is a contradiction to \( \alpha \) being sustainable.

(ii) The proof for the second claim follows the same logic. Let \( \alpha (K, L) \) be a sustainable pure agreement with isolated subsets \( K, L \) of policy issues. To link policy issues affects outcome paths only after a simple defection within \( K \) or \( L \). In this latter case, the severity of the punishment increases by linking policy issues which can only raise the cost of a simple defection and leaves benefits unaffected. Formally,
\[
c_{\alpha'(K,L)} (J, L) \geq c_{\alpha(K,L)} (J, L),
b_{\alpha'(K,L)} (J, L) = b_{\alpha(K,L)} (J, L) = b (J, L).
\]
which implies the second claim. □

**Proof of Lemma 2.** We proceed by contradiction. Suppose that \( j \) and \( k \) are substitutes, and that \( b (J, \{ j, k \}) \geq b (J, \{ j \}) + b (J, \{ k \}) \). Substituting from the definitions one obtains
\[
u_{J} (p (J, \{ j, k \})) - u_{J} (\varphi) \geq u_{J} (p (J, \{ j \})) + u_{J} (p (J, \{ j \})) - 2u_{J} (\varphi)
\]
where, say, \( p (J, \{1, 2\}) = v_{J} (u_{J1} (D, C), u_{J2} (D, C), u_{J3} (C, C), ..., u_{Jn} (C, C)) \). One obtains
\[
u_{J} (p (J, \{ j, k \})) + u_{J} (\varphi) \geq u_{J} (p (J, \{ j \})) + u_{J} (p (J, \{ j \}))
\]
which is the definition of supermodularity for the function \( u_{J} \) with respect to arguments \( j \) and \( k \). But this cannot be satisfied, since a differentiable function is supermodular if and only if the cross derivative is weakly positive for all its arguments (a simple proof of this is in Fudenberg and Tirole 1991, p. 490, footn. 14), which contradicts the assumption that issues \( j \) and \( k \) are substitutes.

Analogously, suppose that \( j \) and \( k \) are substitutes and that \( c_{\alpha} (J, \{ j, k \}) \leq c_{\alpha} (J, \{ j \}) + c_{\alpha} (J, \{ k \}) \). First note that for the given assumptions on agreement pattern \( \alpha \) we have
\[
V_{\alpha} (J, L) = \frac{\delta_{J} u_{J} (p_{N \setminus L})}{1 - \delta_{J}}
\]
where, say, \( p_{N \setminus \{1,2\}} = v_{J} (u_{J1} (D, D), u_{J2} (D, D), u_{J3} (C, C), ..., u_{Jn} (C, C)) \). Substituting from the definitions, simplifying common factors, and compacting notation we obtain
\[
\delta_{J} [U_{J} (\alpha) - V_{\alpha} (J, \{ j, k \})] \leq \delta_{J} [U_{J} (\alpha) - V_{\alpha} (J, \{ j \})] + \delta_{J} [U_{J} (\alpha) - V_{\alpha} (J, \{ k \})] \Leftrightarrow \nou_{J} (p_{N \setminus \{j\}}) + u_{J} (p_{N \setminus \{k\}}) \leq u_{J} (\varphi) + u_{J} (p_{N \setminus \{j,k\}})
\]
which is again the definition of supermodularity for the function \( U \) with respect to arguments \( j \) and \( k \), contradicting the assumption that \( j \) and \( k \) are substitutes.

The proof of the second statement (complementarity) is fully analogous. \( \square \)

**Assumptions made in Section 4:** The structure in section 4 is more specific than the general model. Here, we explain in more detail the additional assumptions. (i) Let \( \Gamma \) be a policy stage game that can be separated into \( i = 1, \ldots, n \) different policy games with payoff functions \( u_{Ji} \). Aggregated stage game payoff for country \( J \) is evaluated by a strictly monotone function \( v_J \), that is

\[
u_J(p) = v_J(u_{J1}(p_{A1}, p_{B1}), \ldots, u_{Jn}(p_{An}, p_{Bn})).
\]

(ii) Punishments are specified in the most simple way that allows for the analysis of policy issue linkage. We assume that any violation \( (J, L) \) is answered by **optimal punishment on the punishment set** \( P_\alpha(J, L) \) with \( L \subset P_\alpha(J, L) \), i.e. the defector obtains continuation utility

\[
V_{\alpha}(J, L) = \frac{\delta u_J(p_{C,\alpha}(L))}{1 - \delta_J},
\]

where \( u_J(p_{C,\alpha}(L)) = v_J(z_{J1}, \ldots, z_{Jn}) \) with

\[
z_{Ji} = \begin{cases} 
 u_{Ji}(D, D) & \text{for } i \in P_\alpha(J, L) \\
 u_{Ji}(C, C) & \text{for } i \in C_\alpha(J, L) = N \setminus P_\alpha(J, L)
\end{cases}.
\]

In case of policy issue linkage \( P_\alpha(J, L) \) can be strictly larger than the defection set \( L \). The only relevant description of agreement pattern \( \alpha \) is then the ”punishment correspondence” \( P_\alpha(J) : \varnothing(N) \setminus \emptyset \rightarrow \varnothing(N) \setminus \emptyset \) that specifies for each country \( J \). In this case the cost of a simple defection on \( L \) is given by

\[
c^J_\alpha(L) = \frac{\delta_J}{1 - \delta_J} \left[ u_J(\varnothing) - u_J(p_{C,\alpha}(L)) \right]
= \frac{\delta_J}{1 - \delta_J} \left[ v_J(u_{J1}(C, C), \ldots, u_{Jn}(C, C)) - v_J(z_{J1}, \ldots, z_{Jn}) \right].
\]

**Hint how to verify the claim at the end of section 4.1.:** A little calculation yields

\[
\delta_J(\alpha(1, 2)) = \max \left\{ \frac{b_{j1} - c_{j1}}{b_{j1} - d_{j1}}, \frac{b_{j2} - c_{j2}}{b_{j2} - d_{j2}}, \frac{b_{j1} - c_{j1} + b_{j2} - c_{j2}}{b_{j1} - d_{j1} + b_{j2} - d_{j2}} \right\}
= \max \left\{ \frac{b_{j2} - c_{j2}}{b_{j1} - d_{j1} + b_{j2} - d_{j2}} \right\}
\]

\[
\delta_J(\alpha^l(1, 2)) = \max \left\{ \frac{b_{j1} - c_{j1}}{b_{j1} - d_{j1} + c_{j2} - d_{j2}}, \frac{b_{j2} - c_{j2}}{b_{j2} - d_{j2} + c_{j1} - d_{j1}} \right\}
= \max \left\{ \frac{b_{j2} - c_{j2}}{b_{j1} - d_{j1} + b_{j2} - d_{j2}} \right\}
\]

Consider for example parameters \( b_{j1} = 4, c_{j1} = 3, d_{j1} = 2, b_{j2} = 4, c_{j2} = 3, d_{j2} = 1 \) yielding the property \( \frac{2}{5} = \delta_J(\alpha^l(1, 2)) < \delta_J(\alpha(1, 2)) = \frac{1}{2} \). \( \square \)
Proof of Proposition 2. First, we describe punishments for the renegotiation proof agreement pattern. Any violation \((J, L)\) is punished as follows: The defecting country has to cooperate forever after the violation period while the other (suffering) country \(-J\) is allowed to defect in some periods \(t \in \mathbb{N} \setminus S_\lambda\) where \(S_\lambda\) is defined by 

\[
\lambda = (1 - \delta_J) \sum_{t \in S_\lambda} \delta_J^t.
\]

The difficult part of the proof is done in the technical Lemma 2 below, namely, to show that there exists such a subset of integers. At the same time the proof of Lemma 2 will demonstrate why this is precisely true for \(\delta \geq \frac{1}{2}\) and only approximately true for \(\delta_J < \frac{1}{2}\).

If it exists this punishment has the nice theoretical property that if, say, country \(A\) defects it yields minmax continuation payoff

\[
\sum_{t \in \mathbb{N} \setminus S_\lambda} \delta_J^t u_A (p_A^0) + \sum_{t \in S_\lambda} \delta_J^t u_A (\varphi) = \frac{(1 - \lambda) u_A (p_A^0)}{1 - \delta_J} + \frac{\lambda u_A (\varphi)}{1 - \delta_J} = \frac{u_A (\omega)}{1 - \delta_J}
\]

to this defecting country \(A\) which is exactly what country \(A\) could guarantee itself from there by ceasing the relationship and defect forever. This equals the continuation payoff of the grim trigger punishment where both countries forever cease to cooperate with respect to all policy issues. Anything below this payoff cannot be an equilibrium of the continuation game. The punishing country \(B\), however, obtains payoff

\[
\frac{(1 - \lambda) u_B (p_A^0)}{1 - \delta_J} + \frac{\lambda u_B (\varphi)}{1 - \delta_J} > \frac{u_B (\varphi)}{1 - \delta_J}
\]

which is strictly better than without punishment at all. Obviously, country \(B\) would not agree to a proposal to renegotiate – i.e. to switch to another equilibrium of the continuation game. Clearly this latter property does not hold for the grim trigger agreement pattern where both countries suffer if the punishment is implemented. By this construction, the present sustainable and renegotiation proof agreement pattern is not only an efficient equilibrium if both countries cooperate but it remains to be efficient if a country defects. □

To state and prove the lemma let \(0 < \delta < 1\), \(S \subset \mathbb{N}\). Introduce the notation \(\lambda (S, \delta) = (1 - \delta) \sum_{t \in S_\lambda} \delta^t\) and \(P_\delta = \{ \lambda (S, \delta) | S \subset \mathbb{N} \}\). Let further \(Q_\delta\) denote the topological closure of the complement \(\mathbb{C}P_\delta := [0, 1] \setminus P_\delta\).

**Lemma 2** \(P_\delta = [0, 1]\) for \(\delta \geq \frac{1}{2}\) and \(Q_\delta = [0, 1]\) for \(\delta < \frac{1}{2}\).

**Proof.** (i) Let first \(\delta \geq \frac{1}{2}\). Then, \(\delta^{n-1} \leq \frac{\delta^n}{1 - \delta}\) for all \(n \in \mathbb{N}\). We have to show that for any \(\lambda \in [0, 1]\) there exists \(S \subset \mathbb{N}\) with \(\lambda = \lambda (S, \delta)\). This is obvious for \(\lambda = 0\) and \(\lambda = 1\), therefore assume \(0 < \lambda < 1\). Let \(n_1\) be minimal such that \(\frac{\delta^{n_1}}{1 - \delta} \leq \lambda\) and \(n_1 \neq 0\). This implies

\[
\lambda < \frac{\delta^{n_1 - 1}}{1 - \delta} \leq \frac{\delta^{n_1}}{(1 - \delta)^2} \Rightarrow
\]

\[
\frac{\lambda}{\delta^{n_1}} < \frac{1}{(1 - \delta)^2}.
\]

\^[14]The proof of this lemma was contributed by Werner Blonski.\]
If $n_1 = 0$ this holds by assumption. Let $m_1 \in \mathbb{N}$ be maximal such that

$$\frac{1 - \delta^{m_1}}{(1 - \delta)^2} \leq \frac{\lambda}{\delta^{m_1}}.$$

Hence, $m_1 \neq 0$ and

$$\frac{\lambda}{\delta^{m_1}} < \frac{1 - \delta^{m_1 + 1}}{(1 - \delta)^2} \Rightarrow$$

$$\frac{\lambda}{\delta^{m_1}} - \frac{1 - \delta^{m_1}}{(1 - \delta)^2} < \frac{\delta^{m_1}}{1 - \delta} \Rightarrow$$

$$\frac{\lambda - \delta^{m_1}(1 - \delta^{m_1})}{(1 - \delta)^2} < \frac{\delta^{m_1 + 1}}{1 - \delta}.$$

Denote by $p_1 = \frac{\delta^{m_1}(1 - \delta^{m_1})}{(1 - \delta)^2}$ and $\lambda_1 = \lambda - p_1$, then $p_1 = \frac{1}{1 - \delta} \sum_{i=1}^{m_1} \delta^i$ with $\lambda_1 = \lambda - p_1 < \frac{\delta^{m_1 + m_2}}{(1 - \delta)^2}$. We apply the same procedure to $\lambda_1$ and obtain $\lambda_2 = \lambda_1 - p_2 < \frac{\delta^{m_2 + m_3}}{(1 - \delta)^2}$, again $\lambda_3 = \lambda_2 - p_3$ and so forth. If $\lambda_k = 0$, this construction ends with $\lambda = \sum_{i=1}^{k} p_i \in P_k$. Otherwise we get $\lambda = \sum_{i=1}^{\infty} p_i \in P_k$ with $\lambda = \lambda(S, \delta)$ where $S \subset \mathbb{N}$ is composed by the intervals $[n_i, n_i + m_i - 1] \subset \mathbb{N}$ with $n_i + m_i < n_{i+1}$.

(ii) Next, consider $\delta < \frac{1}{2}$. In this case $\delta^{n+1} < \delta^n$ for all $n \in \mathbb{N}$. Define $J_n := \left[ \frac{\delta^n}{1 - \delta}, \frac{\delta^n}{(1 - \delta)^2} \right]$ (closed interval), open interval $K_n = \left( \frac{\delta^{n+1}}{1 - \delta}, \frac{\delta^n}{(1 - \delta)^2} \right)$, $J = \bigcup J_n$ and $K = \bigcup K_n$. Then, $J \cap K = \emptyset$. Fix $\delta$ and denote $\lambda_S \equiv \lambda_S(\delta)$. Let $\lambda_S = \frac{1}{1 - \delta} \sum_{i \in S} \delta^i \in K_n$ and $S_n = \{i + 1, i + 2, \ldots \}$. Hence, $\lambda_{S_n} = \frac{\delta^{n+1}}{(1 - \delta)^2}$ and for $\lambda_S \leq \lambda_{S_n}$ follows $S \subset S_n$. Therefore, $\lambda_S \notin K_n$ and $K_n \subset \mathbb{C} P_k$. Let further $Z_n = \{0, 1, \ldots, n - 1\}$ and $K_{S,n} = \lambda_S + K_n$ with $S \subset Z_n$ be a system of open intervals. For $S = \{n_1\}$ with $n_1 < n_2 < \ldots$ we have $K_{S,n} \subset J_{n_1}$. For another interval $K_{S',n'}$ of this type is $K_{S,n} \cap K_{S',n'} \subset J_{n_1} \cap J_{n_1} = \emptyset$ if $n_1 \neq n_1'$. Conversely, for $K_{S,n} \cap K_{S',n} \neq \emptyset$ follows $n_1 = n_1'$. Omission of the first expression \(\delta_{i}^{n_1} \frac{1}{1 - \delta}\) yields $n_2 = n_2'$ etc., and finally $n = n'$. Therefore the system of intervals $K_{S,n}$ is pairwise disjoint. We will show that $K_{S,n} \subset \mathbb{C} P_k$. Let $\lambda_T \in \mathbb{C} P_k$, that is $\lambda_T = \lambda_S + \lambda$ where $\lambda \in K_n$. We conclude similarly that $S \subset T$ and thereby $\lambda_T - S \in K_n$. This, however, contradicts $K_n \subset \mathbb{C} P_k$. Next we show $\mathbb{C} P_k = \bigcup K_{S,n}$. To see this let $\lambda \in (0, 1] \setminus \bigcup K_{S,n}$. Then there exists a minimal $n_2$ with $\frac{\delta^{n_2}}{1 - \delta} \leq \lambda - \frac{\delta^{m_1}}{1 - \delta}$. Continuation of this construction yields a set $T \subset \mathbb{N}$ with $\lambda_T = \lambda$. The second claim of the lemma $Q_\delta = [0, 1]$ means, that any neighborhood of $\lambda_T \in P_k$ has a non-empty intersection with $\mathbb{C} P_k$. Let $\lambda_T = \lambda_S + \lambda_{T-S}$, $S = T \cap Z_n$ and $T - S = \{m_i\}$. In this case is $m_1 \geq n$ and $\lambda_{T-S} \in J_{m_1} = \left[ \frac{\delta^{m_1}}{1 - \delta}, \frac{\delta^{m_1}}{(1 - \delta)^2} \right]$ with $\frac{\delta^{m_1}}{(1 - \delta)^2} - \frac{\delta^{m_1}}{1 - \delta} = \frac{\delta^{m_1 + 1}}{(1 - \delta)^2}$. For any $\varepsilon > 0$ we choose $S$, such that $\frac{\delta^{m_1 + 1}}{(1 - \delta)^2} < \varepsilon$. Then, $[\lambda_T - \varepsilon, \lambda_T + \varepsilon]$ has a non-empty intersection with $K_{S,m_1 - 1}$ and $K_{S,m_1}$ and thereby with $\mathbb{C} P_k$. This implies $Q_\delta = [0, 1]$. Finally, if $S$ is finite choose $K_{S,n}$ with $n$ large enough such that $K_{S,n} \subset [\lambda - \varepsilon, \lambda + \varepsilon]$ where $\lambda = \lambda(S, \delta)$. This completes the proof. ■
References


